Def: Let X be a set. A topology on X is a collection τ of subsets of X satisfying

1) $\tau \supseteq \{X, \phi\}$

2) If $\{U_\alpha\}_{\alpha \in I} \in \tau$ then $\bigcup_{\alpha \in I} U_\alpha \in \tau$

3) If $U_1, \dots, U_n \in \tau$ then $\bigcap_{k=1}^{n} U_k \in \tau$.

We call the elements of τ open sets and (X, τ) is a topological space.

Last class we saw that the open sets in a metric space satisfy 1) & 2). They also satisfy 3) and hence form a topology on X.

Pf: Suppose $U_1, \dots, U_n$ are open sets in a metric space. Let $x \in \bigcap_{k=1}^{n} U_k$. Since each U_k is open we can find radii r_k such that $B_{r_k}(x) \subseteq U_k$. Let $r = \min_{k=1 \dots n} r_k$. Then for each

k $\quad B_r(x) \subseteq B_{r_k}(x) \subseteq U_k$ and hence $B_r(x) \subseteq \bigcap_{k=1}^{n} U_k$.

Every metric induces a topology on a set.

Observe that d_1, d_2 and d_∞ all induce the same topology on $\mathbb{R}^2$.

Question: Is every topology the topology induced by some metric?

Two trivial and fundamental topologies

1) Largest possible topology on X.

$\tau = \mathcal{P}(X)$

Singletons are open.

"discrete topology"

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$$\{x\} = B_{1/2}(x)$$

2) Smallest possible topology X

$\tau = \{\phi, X\}$

"indiscrete topology"

If X has more than one element then it is not induced by any metric.

Suppose $x, y \in X$ $x \neq y$ and d is a metric on X.

Let $r = d(x,y)$. Since $x \neq y$, $r > 0$.

Consider $B_{r/2}(x)$. This ball is open, contains x and excludes y. Since $B_{r/2}(x) \neq X$ and $\neq \phi$, the metric does not induce the indiscrete topology

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Def: A topological space is <u>metrizable</u> if its topology is induced by some metric.

As for metric spaces, a set $V \subseteq X$ is closed if $V^c = X \setminus V$ is open

In every space both X and $\emptyset$ are closed.

De Morgan's Laws

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

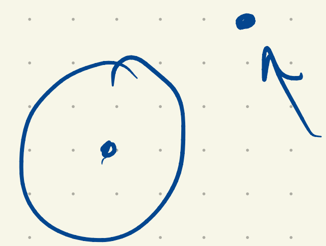
$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

Exercise: Use these to show that an arbitrary intersection of closed sets is closed and a finite union of closed sets is closed.

E.g. In a metric space define $\overline{B}_r(x) := \{ y \in X : d(x,y) \leq r \}$.

Execise: For all $r \geq 0$, $\overline{B}_r(x)$ is a closed set.

(triangle inequality)



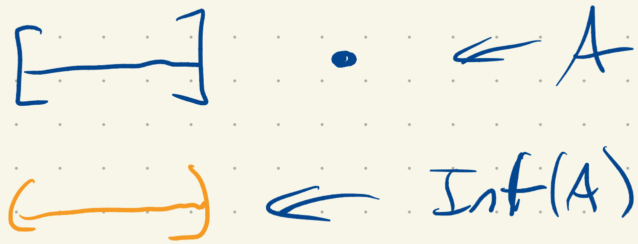
e.g. $[-1,1]$ is closed in $\mathbb{R}$. $\overline{B}_1(0)$

Vaguely: topologies encode a notion of "nearness" and "adjacency"!

Def: Let $A \subseteq X$ (a topological space).

The <u>interior</u> of A $(\mathrm{Int}(A))$ is the union of all open sets contained in A.

The closure of A, $\overline{A}$, is the intersection of all closed sets containing A.



Observe that the interior of A is open.

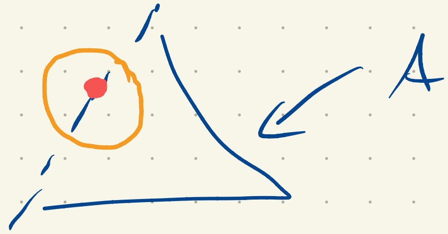
Exercise: it is the largest open set contained in A.

largest implies any open set in A is contained in $\mathrm{Int}\, A$.

Observe that the closure of a set A is closed and it is the smallest closed set that contains A.

Def: A point $x \in X$ is a <u>contact point</u> of A if whenever U is an open set containing x,

$$U \cap A \neq \emptyset.$$



$(-1,1) \subseteq \mathbb{R}$

$[-1,1] \leftarrow$ contact points

$\mathbb{Q} \subseteq \mathbb{R}$ contact points: $\mathbb{R}$

Proposition: Given $A \subseteq X$, $\bar{A}$ is precisely the set of contact points of A.

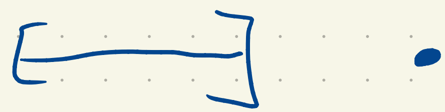
Pf: Let A' denote the contact points of A.

Suppose $x \notin A'$. Then there exists an open set U such that $x \in U$ but $U \cap A = \phi$. Let $V = U^c$. Then $x \notin V$, V is closed and $V \supseteq A$. Since $\bar{A} \subseteq V$, $x \notin \bar{A}$.

Suppose $x \notin \bar{A}$. Then there exists a closed set V with $A \subseteq V$ but $x \notin V$. Let $U = V^c$. Then $x \in U$, U is open, and $U \cap A = \phi$. So $x \notin A'$. $\square$

Contact points are points that are either in A or are adjacent to A.

Def: Given $A \subseteq X$, a point $x \in X$ is a limit point of A if every open set containing x contains a point from A that is different from x.



Def: The exterior of A is $(\overline{A})^c$.

We write this as $\text{Ext}(A)$.

Note: $x \in \text{Ext}(A) \iff \exists\, U$, open, $x \in U$, $U \cap A = \emptyset$.

What are the points that are adjacent to both A and A^c?

This is the *boundary* of A, $\partial A = \overline{A} \cap \overline{A^c}$.

not text's def.

Prop 2.8 (Unwieldy collection of facts)