On $\mathbb{R}^2$ the l_1, l_2 and l_∞ distances all determine the same convergent sequences.

Lemma: Suppose d and $\hat{d}$ are metrics on X such that there exists some $C > 0$ with

$$d(x,y) \leq C\,\hat{d}(x,y) \quad \forall x,y \in X.$$

Then if $x_n \xrightarrow{\hat{d}} x$ for some sequence then

$$x_n \xrightarrow{d} x.$$

Pf: Easy.

Def: Two metrics d, $\hat{d}$ on X are <u>equivalent</u> if there exist constants $c, C > 0$ such that

$$c\,d(x,y) \leq \hat{d}(x,y) \leq C\,d(x,y) \quad \forall x,y \in X.$$

$$\left[\begin{array}{c} \hat{d}(x,y) \leq C\, d(x,y) \\ d(x,y) \leq \frac{1}{c}\, \hat{d}(x,y) \end{array}\right]$$

Consequently if two metrics are equivalent they determine the same convergent sequences and the same continuous functions into $\mathbb{R}$

Metrics d_1, d_2 and d_∞ are all equivalent

$$d_\infty \leq d_2 \qquad d_2 \leq \sqrt{2}\, d_\infty$$

$$d_\infty \leq d_1 \qquad d_1 \leq 2 d_\infty$$

$$d_\infty(x,y) = \max\left(|x_1-y_1|, |x_2-y_2|\right) \leq |x_1-y_1| + |x_2-y_2| = d_1(x,y)$$

Open Sets

Def: Let (X,d) be a metric space. Given $x \in X$ and $r > 0$ the open ball about x of radius r is

$$B_r(x) = \{ y \in X : d(x,y) < r \}$$

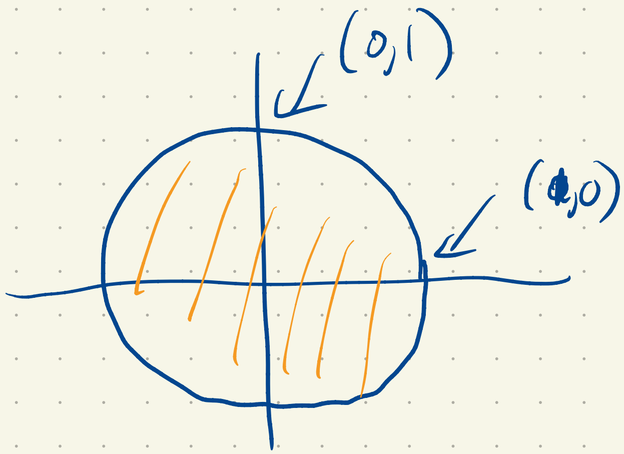
e.g. 1) $(\mathbb{R}, |\cdot|)$

$$B_1(0) = (-1,1)$$

2) $(\mathbb{R}^2, d_2)$

$$B_1(0) = \{ (x,y) : \sqrt{(x-0)^2 + (y-0)^2} < 1 \}$$

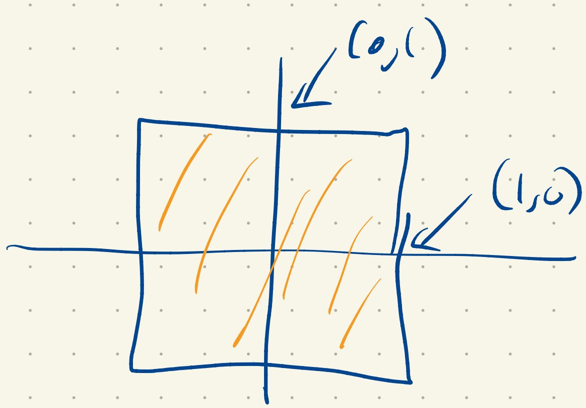
$$= \{(x,y) : x^2+y^2 < 1\}$$



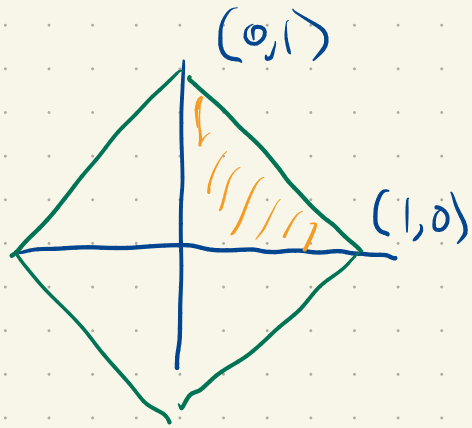
3) $(\mathbb{R}^2, d_\infty)$

$$B_1(0) = \{(x,y) : \max(|x-0|, |y-0|) < 1\}$$

$$= \{(x,y) : \max(|x|, |y|) < 1\}$$

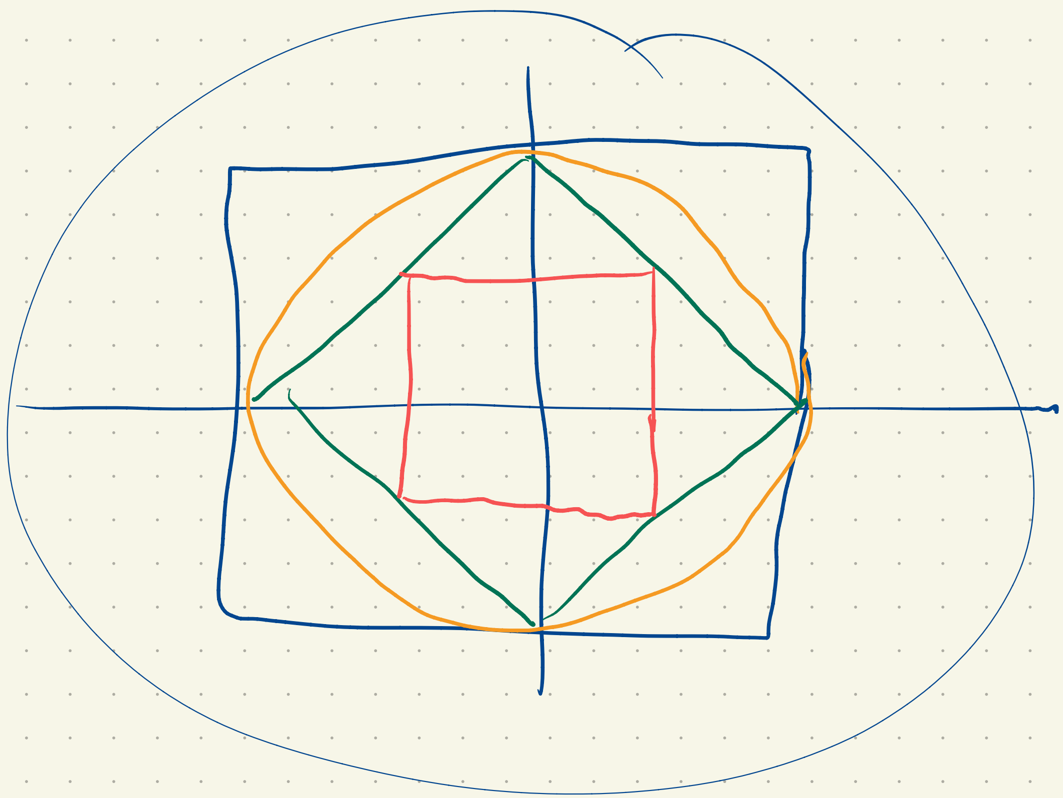


4) $(\mathbb{R}^2, d_1)$ $B_1(0) = \{ (x,y) : |x| + |y| < 1 \}$



$x + y < 1$

$y < 1 - x$

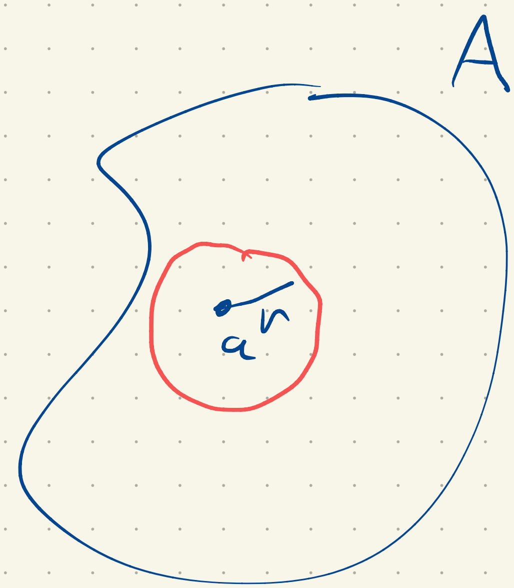


Def. Let (X,d) be a metric space.

A set $A \subseteq X$ is open if

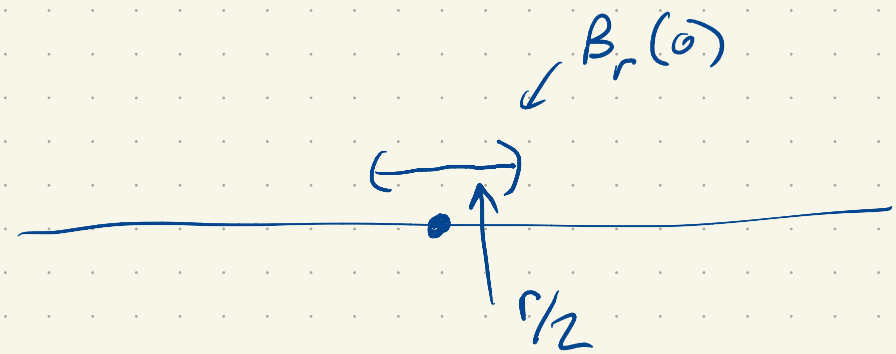
for all $a \in A$ there exists $r > 0$

such that $B_r(x) \subseteq A$



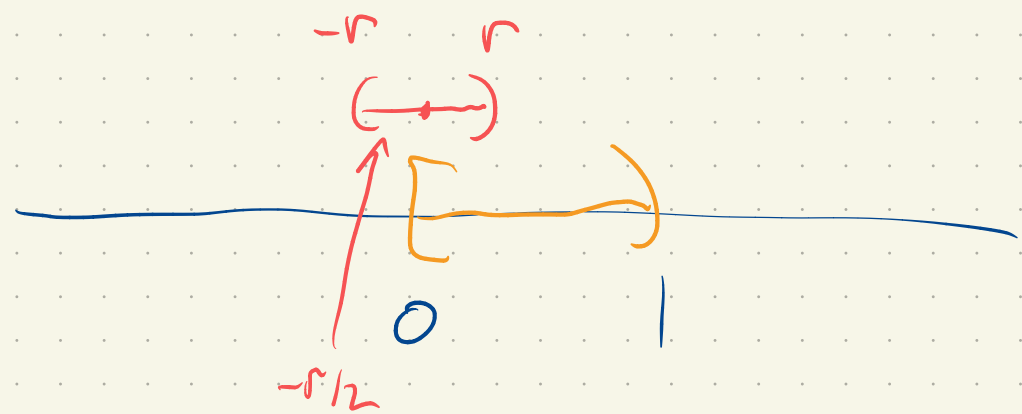
e.g. $\{0\} \subseteq \mathbb{R}$

not open!



$[0,1)$

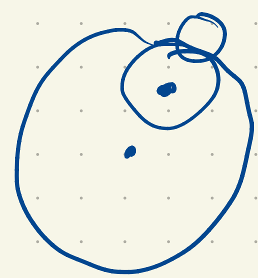
not open!



$(0,1)$ is open

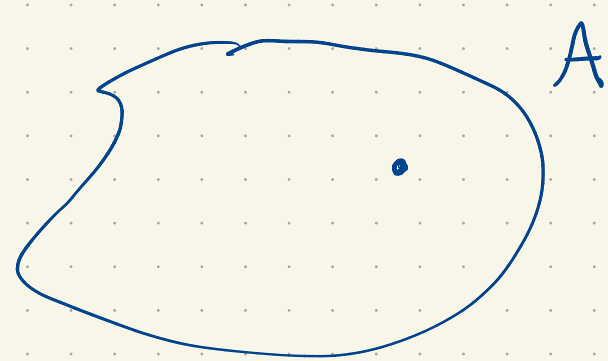
Exercise: Any ball $B_r(x)$ in a metric space is open.

Triangle inequality!



Exercise: Suppose d and $\hat{d}$ are equivalent metrics.

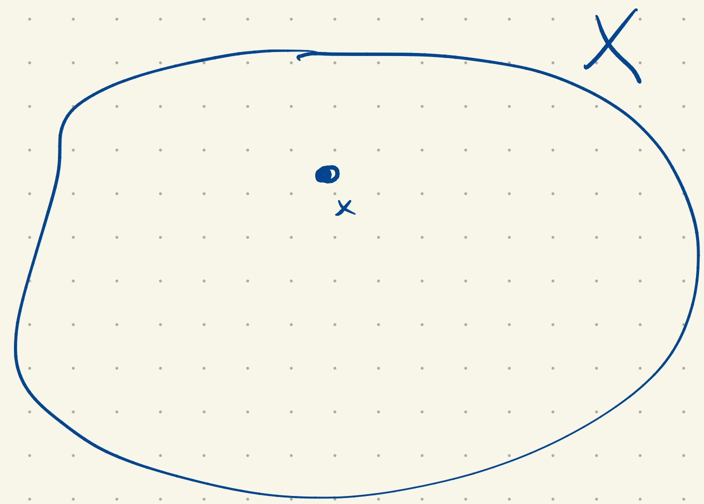
Then a set $A \subseteq X$ is open with respect to d iff it is open w.r.t. $\hat{d}$.



d_1, d_2 and d_∞ on $\mathbb{R}^2$ determine the same open sets.

Def: A set $A \subseteq X$ is *closed* if $A^c \, (= X \setminus A)$ is open.

X is open.



X is closed.

$\emptyset$ is open!

In $\mathbb{R}$ $[-1, 1]$ is closed.

$(-\infty, -1) \cup (1, \infty)$ is open

Lemma: An arbitrary union of open sets in a metric space is open.

$$(1, \infty) = \bigcup_{a>1} (1, a) \quad , \quad (-\infty, -1) = \bigcup_{a>1} (-a, -1)$$

Pf: Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets and let $U = \bigcup_{\alpha \in I} U_\alpha$. Consider some $x \in U$. Then $x \in U_\alpha$ for some α. Since U_α is open there exists $r > 0$ such that $B_r(x) \subseteq U_\alpha \subseteq U$. $\square$

If two metrics are equivalent

1) They determine the same convergent sequences.

2) They determine the same continuous functions into $\mathbb{R}$

3) They determine the same open sets

4) They determine the same closed sets.

From your homework.

Suppose d_1 and d_2 are two metrics on X.
Then the following are equivalent (TFAE)

1) For all sequences $\{x_n\}$, if $x_n \xrightarrow{d_2} x$ then $x_n \xrightarrow{d_1} x$.

2) For all functions $f: X \to \mathbb{R}$, if f is continuous with respect to d_1 then f is continuous with respect to d_2

3) For all $U \subseteq X$, if U is open w.r.t. d_1 then U is open w.r.t. d_2

4) For all $V \subseteq X$, if V is closed wrt d_1 then V is closed w.r.t. d_2.

If any of 1) – 4) is an iff for d_1 and d_2 then all of 1) – 4) are iff.

One could hope that maybe equivalence classes of metrics are the right underlying object of study

But $\hat{d}(x,y) = \left| \int_x^y e^s \, ds \right| = \left| e^y - e^x \right| \cancel{\leq C |x-y|}$

is a metric on $\mathbb{R}$ that is not equivalent to the usual metric. But, shortly, we'll have a good tool to show that U is open in $\mathbb{R}$ with respect to $\hat{d}$ iff it is open wrt the usual metric