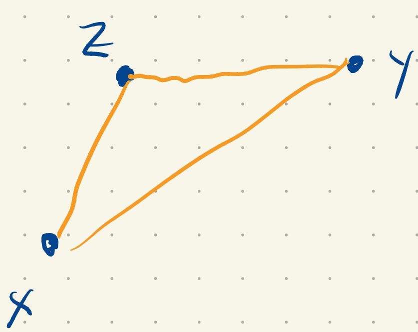
Metric Spaces

Recall

Def: A metric space is a set X together with

$d: X \times X \to \mathbb{R}$ satisfying

1) $d(x,y) \geq 0 \quad \forall x,y \in X \quad \left(d(x,y) = 0 \iff x = y \right)$
2) $d(x,y) = d(y,x) \quad \forall x,y \in X$
3) $\underbrace{d(x,y) \leq d(x,z) + d(z,y)}_{\text{triangle inequality}}, \quad \forall x,y,z \in X$



e.g

1) $\mathbb{R}$ $\quad d(x,y) = |x-y|$ $\quad$ (Exercise!)

2) $\mathbb{R}^2$ $\quad \|x\|_2 = (x_1^2 + x_2^2)^{1/2}$ $\quad$ (l_2 norm)

($x = (x_1, x_2)$)

$$d_2(x,y) = \|x-y\|_2$$

(Exercise!)

Cauchy-Schwartz ineq.

$$x \cdot y = \|x\|_2 \|y\|_2 \cos\theta$$

3) $\mathbb{R}^2$ $\quad \|x\|_1 = |x_1| + |x_2|$

$$d_1(x,y) = \|x-y\|_1 \quad \text{(Exercise!)}$$

4) $\mathbb{R}^3$ $\quad \|x\|_\infty = \max(|x_1|, |x_2|)$

$$d_\infty(x,y) = \|x-y\|_\infty \quad \text{(Exercise!)}$$

5) X is any set

$$d_0(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

(discrete metric)

6) $X = C[0,1]$, continuous functions on $[0,1]$

$\hookrightarrow [0,1] \to \mathbb{R}$

$$\|f\|_2 = \left[\int_0^1 |f(x)|^2 dx\right]^{1/2}$$ L^2 norm

$$d_2(f,g) = \|f-g\|_2$$ L^2 distance

$\|x\|_\infty = \max(|x_1|, |x_2|)$

7) $X = C[0,1]$

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

$$d_\infty(f,g) = \|f-g\|_\infty$$

Let (X, d) be a metric space.

We say $\{x_n\}$ converges to x in X if

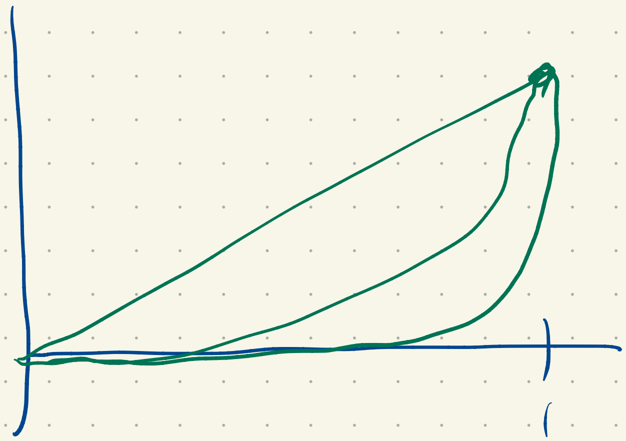
$$\lim_{n\to\infty} d(x_n, x) = 0.$$

[for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if

$n \geq N$ $\quad |0 - d(x_n, x)| < \varepsilon$

$d(x_n, x) < \varepsilon$]

$f_n(x) = x^n \qquad (C[0,1])$



$f_n \overset{?}{\to}$

L^2

$f_n \to 0$ w.r.t. L^2 distance

$$d_2(f_n, 0)^2 = \int_0^1 (x^n)^2 \, dx = \frac{1}{2n+1} \to 0$$

$$\lim_{n \to \infty} d_2(f_n, 0) = 0 \Rightarrow f_n \xrightarrow{L^2} 0$$

$d_\infty(f_n, 0) = 1$ for all n. $\{f_n\}$ does not converge w.r.t. ∞ norm

Def: Suppose $f: (X, d_X) \to (Y, d_Y)$.

We say f is <u>continuous</u> if whenever $x_n \to x$ in X,

$f(x_n) \to f(x)$ in Y.

E.g. $f: \mathbb{R} \to \mathbb{R}$, continuity in Math 401 sense.

E.g. (X, d_X) any metric space. $y \in X$ fixed.

$$f(z) = d(z, y)$$

f is continuous. Exercise!

[Triangle inequality!]

Different metrics on a set provide different notions of continuity.

$$X = C[0,1]$$

$$M : X \to \mathbb{R}$$

$$M(f) = \max_{x \in [0,1]} f(x)$$

Is M continuous?

With respect to L^2 distance?

$$f_n = x^n$$

$M(f_n) = 1$ $\qquad$ $f_n \to 0$

$M(0) = 0$ $\qquad$ $M(f_n) \to M(0)$? Nope.

Exercise: M is continuous w.r.t. L^∞ distance.

Note: The metric shows up only indirectly in the notion of continuity via the notion of convergence.

If two different metrics determine the same notion of convergence then they determine the same continuous functions.

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d_1 $\qquad\qquad$ $d_2 = 5\,d_1$

These determine the same convergent sequences

On $\mathbb{R}^2$ the l_1, l_2 and l_∞ distances all determine the same convergent sequences.