1. Exercise 0.1 (Solution by John Gimbel)

If a and b are even integers, then so is $a + b$.

Solution:
Let a and b be even integers. Then there exist integers j and k such that $a = 2j$ and $b = 2k$. But then

$$a + b = 2j + 2k = 2(j + k). \tag{1}$$

Since $j + k \in \mathbb{Z}$, $a + b$ is even.

2. Exercise 0.2 (Solution by Jill Faudree)

Let X be a set.

a) An intersection of topologies on X is a topology on X.

b) A union of topologies on X need not be a topology.

Solution, part a:
Let $\{\tau_\alpha\}$ be a family of topologies and let $\tau = \cap_\alpha \tau_\alpha$. Observe that $\varnothing$ and X belong to τ as they belong to each τ_α.

Suppose $\{U_\beta\}$ is a family of sets in τ and let $U = \cup_\beta U_\beta$. Fix α and observe that each $U_\beta \in \tau_\alpha$. Since τ_α is a topology, $U \in \tau_\alpha$. Since α is arbitrary, $U \in \cap \tau_\alpha = \tau$.

The proof that a finite intersection of sets in τ belongs to τ is essentially similar.

Solution, part b:
Let $X = \{1, 2, 3\}$. Let $\tau_1 = \{\varnothing, \{1\}, X\}$ and let $\tau_2 = \{\varnothing, \{2\}, X\}$. It is easy to see that these are topologies. Let $T = \tau_1 \cup \tau_2 = \{\varnothing, \{1\}, \{2\}, X\}$. Observe that T is not closed under taking unions as $\{1\}$ and $\{2\}$ are elements of T but $\{1, 2\}$ is not.

3. Exercise 0.3 (Solution by Elizabeth Allman)

Let X be a metric space. Showt that the collection of open balls in X forms the basis of a topology.

Solution:
We start with a technical lemma.

Lemma A: Suppose $B_1 = B_{r_1}(x_1)$ and $B_2 = B_{r_2}(x_2)$ are open balls in X an $x_3 \in B_1 \cap B_2$. Then there is an $r > 0$ such that $B_r(x_3) \subseteq B_1 \cap B_2$.

Proof. Let $r = \min(r_1 - d(x_3, x_1), r_2 - d(x - 2, x_2)$ and observe that $r > 0$. Now suppose

$z \in B_r(x_3)$. The triangle inequality implies

$$\begin{aligned} d(x_1, z) &\le d(x_1, x_3) + d(x_3, z) \\ &< d(x_1, x_3) + r \\ &\le d(x_1, x_3) + (r_1 - d(x_3, x_1)) \\ &= r_1 \end{aligned}$$

Hence $z \in B_{r_1}(x_1) = x_1$. Similarly $z \in B_2$, and hence $B_r(z) \subseteq B_1 \cap B_2$. □

Let $\mathcal{B}$ be the collection of open balls in X. Fix $x \in X$ and note that $\cup_{r>0} B_r(x) = X$. Hence $\mathcal{B}$ covers X. Moreover, by Lemma A, $\mathcal{B}$ satisfies the refinement property. Hence by the topology construction lemma, $\mathcal{B}$ generates a topology on X, and the open sets are simply the unions of open balls.