1. If t is a unit quaternion then the map $w \mapsto t^{-1}wt$ for w an imaginary quaterion is a rotation. This is the content of the **Rotation by conjugation** section of the text on pages 14–15. Moreover, as we saw in class, $w \mapsto (-t)^{-1}w(-t)$ yeilds the same rotation. That is, the unit quaternions t and $-t$ determine, via conjugation, the same rotation. The question now is: are there any others? The following problems show that no, these are the only ones.

1. Show that if t is a unit quaternion and $t^{-1}wt = w$ for all imaginary quaternions w, then $t = \pm 1$.
2. Now give an easy argument to show that if t_1 and t_2 are two unit quaternions such that $(t_1)^{-1}wt_1 = (t_2)^{-1}wt_2$ for all imaginary unit quaternions w, then $t_1 = \pm t_2$.

2. 1.5.1

3. 1.5.2

4. For $p, q \in \mathbb{H}$, we define $\langle p, q \rangle = \mathrm{Re}(p\overline{q})$, where $\mathrm{Re}(q)$ is the real part of q.

a) Show that if $p \in \mathbb{H}$, then $\langle p, p \rangle = |p|^2$.

b) Using the facts that $\overline{ab} = \overline{b}\,\overline{a}$ and $\mathrm{Re}(\overline{a}) = \mathrm{Re}(a)$ for all $a, b \in \mathbb{H}$, show that $\langle p, q \rangle = \langle q, p \rangle$.

c) For $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, let $Q(x) = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ be its representation as a quaternion. Show that for $x, y \in \mathbb{R}^4$, $x \cdot y = \langle Q(x), Q(y) \rangle$.

d) We say that p and q are orthogonal if $\langle p, q \rangle = 0$. Show that $p \in \mathbb{H}$ is orthogonal to 1 if and only if p is pure imaginary.