Projective transformations

$\mathbb{R}P^1$, $\mathbb{R}P^2$

$\lambda \neq 0$

homog. coords

$(\lambda x, \lambda y)$

$\downarrow$

(x, y)

$x \in \mathbb{R}^+$

$$T(x) = \frac{ax+b}{cx+d}$$

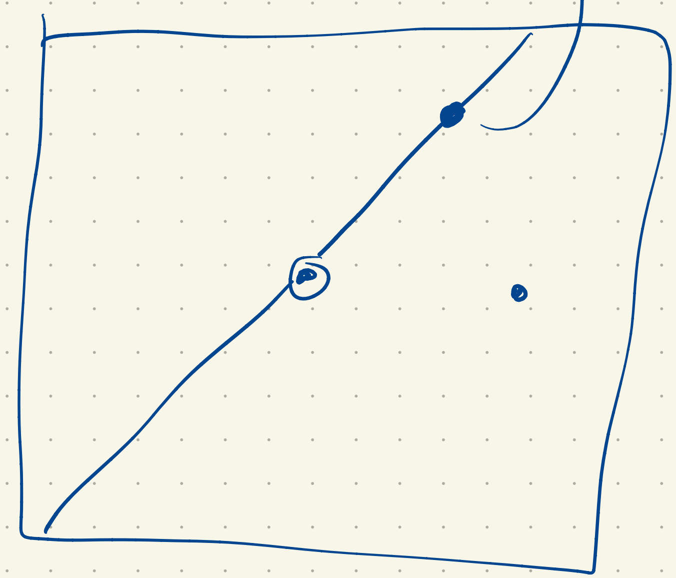
$a, b, c, d \in \mathbb{R}$

$ad - bc \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ w \end{bmatrix}$$

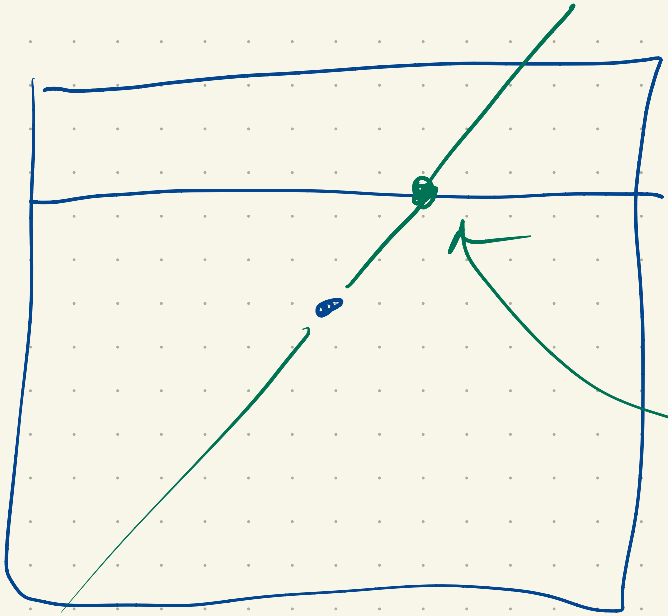
homog coords of p

homog. coords of image



$\mathbb{R}^2$

(x, y, z)



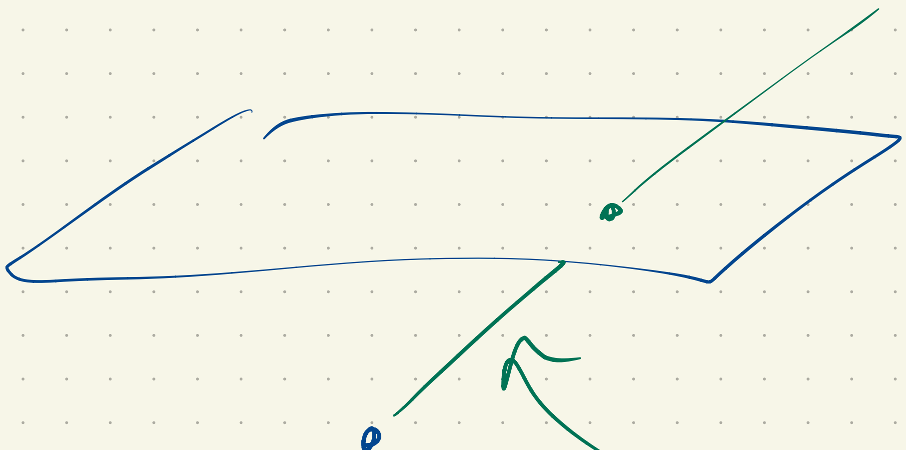
$y=1$

$\underbrace{}_{}$ $\mathbb{R}P^2$

$\mathbb{R}P^1$

$(x,1)$

x, inhomogeneous coords of the projective point.

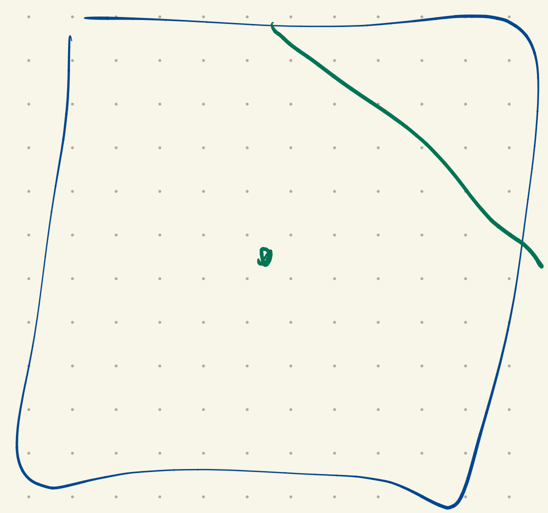


$z=1$

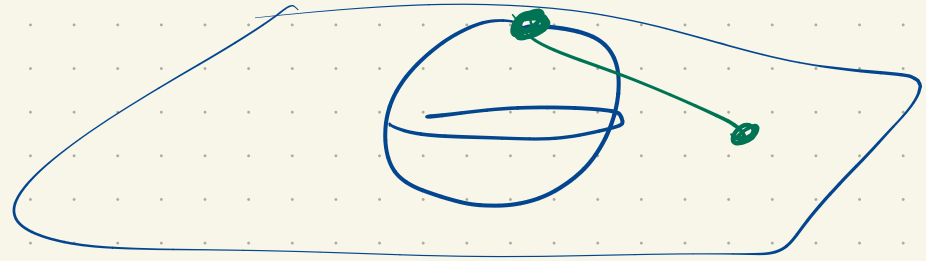
$(x,y,1)$

$\downarrow$

(x,y)



$$[A, B, C]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



Coords in projective geom: ($\mathbb{RP}^2$)

Homogeneous

(x, y, z)

every $p \in \mathbb{RP}^2$ admits homog. coords

loss of uniqueness

whole plane but is hard to visualize

Inhomogeneous

$(x, y, 1)$

omit

most points in $\mathbb{RP}^2$ admit inhomog. coords

we miss a "line at ∞"

unique label

partial picture but is easy to visualize

$[A]$ $\quad A \sim \lambda A$

$\lambda \neq 0$

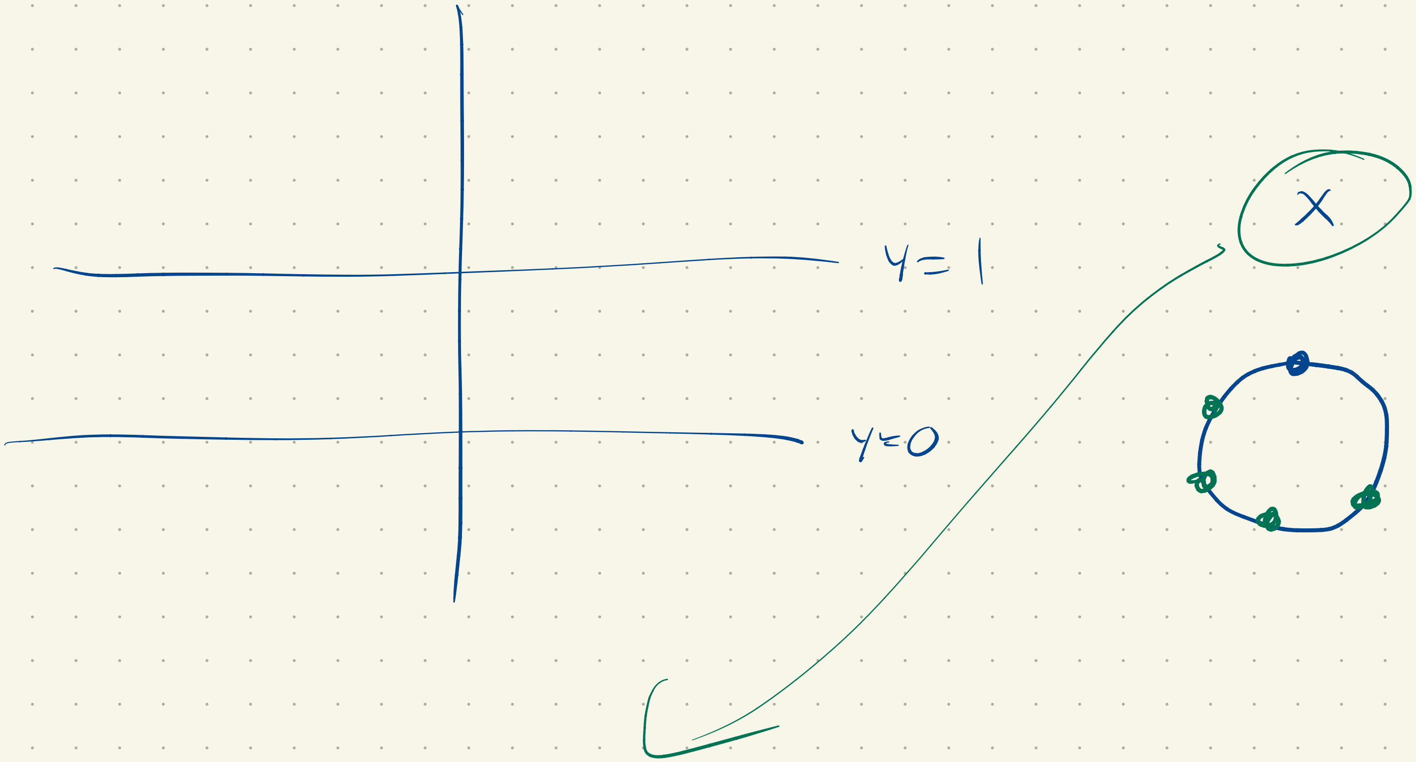
3x3 invertible matrix

$(x,y) \to \left(\frac{ax+by+c}{gx+hy+i}, \frac{dx+ey+f}{gx+hy+i} \right)$

$[A][p] = [Ap]$

scaling by λ

An invariant of $\mathbb{RP}^1$



$$T(x) = \frac{ax+b}{cx+d}$$

Möbius

x_0, x_1, x_2, x_3 four distinct points ($\in \mathbb{R}^+$)

(x_0, x_1, x_2, x_3) is an invariant.

Given distinct $x_1, x_2, x_3 \in \mathbb{R}^+$

$y_1, y_2, y_3 \in \mathbb{R}^+$

There is a unique projective transformation T $T(x_i) = y_i$.

$x_1 \to 1$

$x_2 \to 0$

$x_3 \to \infty$

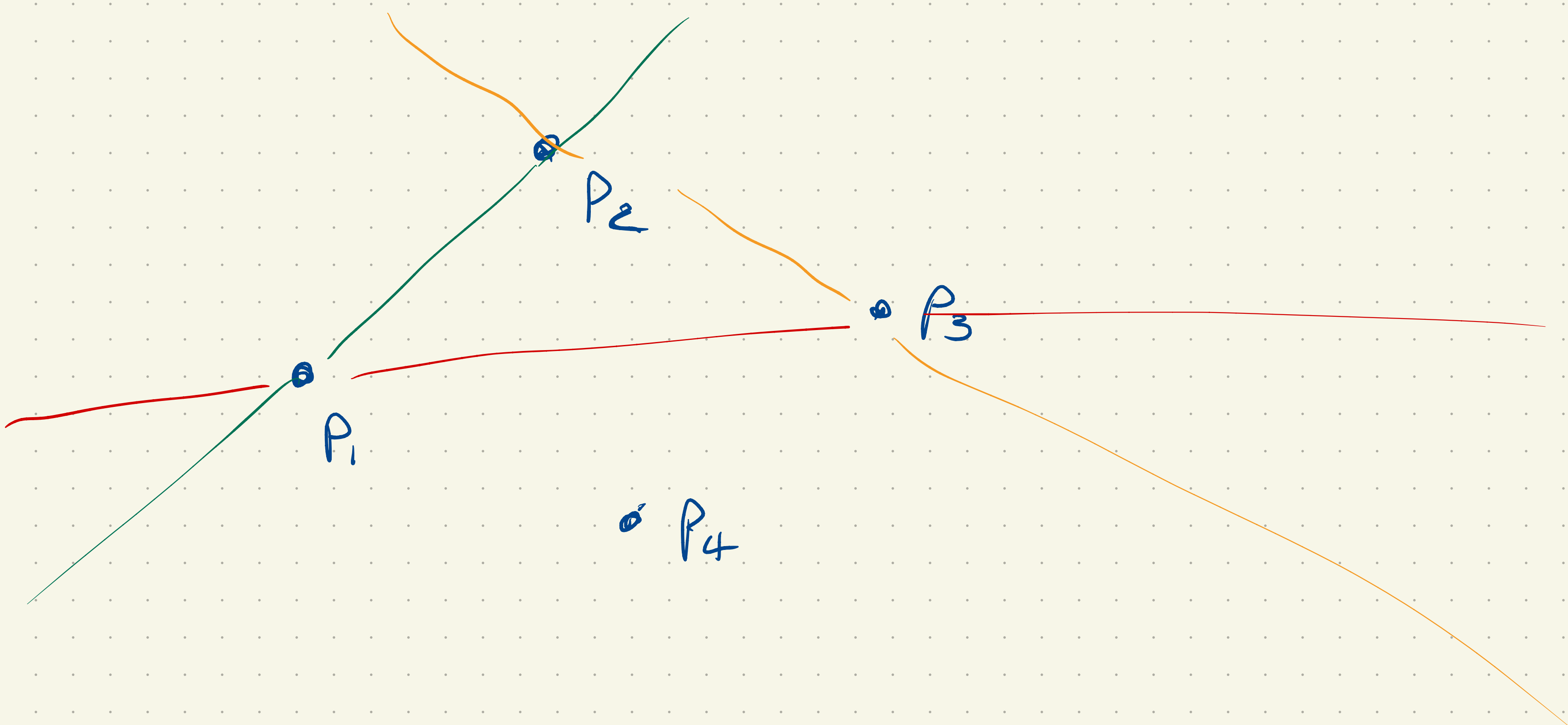
$$T(x) = \frac{(x - x_2)}{(x - x_3)} \frac{(x_1 - x_3)}{(x_1 - x_2)}$$

"Fundamental Theorem of $\mathbb{R}P^1$"

Mobius geometry is really projective geometry of the complex line, $PGL(2, \mathbb{C})$

Fundamental Theorem of $\mathbb{R}P^2$

Def: Let $p_1, p_2, p_3, p_4 \in \mathbb{R}P^2$. We say they are in <u>general position</u> if no three are on a common line.



Thm: Let p_1, p_2, p_3, p_4

q_1, q_2, q_3, q_4

be two sets of four projective points (in $\mathbb{R}P^2$)

in general position.

Then there exists a unique projective transformation

T $\quad T(p_i) = q_i \quad i = 1, 2, 3, 4.$

differences a) 4 points, not 3

b) general position vs distinct.

Pf: (convention: hats imply points in $\mathbb{R}^3$)

nonzero (usable as homogeneous coordinates)

Let $\hat{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\hat{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\hat{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\hat{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

be homogeneous coordinates of points v_2 in $\mathbb{RP}^2$;

It is enough to show that if $P_1 \to P_4$ are in general position then is a transformation taking $v_i \to P_i$.

Consider a matrix $\left[\hat{w}_1 \,\middle|\, \hat{w}_2 \,\middle|\, \hat{w}_3 \right] = T$.

Observe $T(\hat{v}_1) = \hat{w}_1$ and similarly.

So $T(v_i) = p_i$ iff $\hat{w}_i = \lambda_i \hat{p}_i$ for some $\lambda_i \neq 0$,

$i = 1, \ldots, 3$.

Hence our transformation must have the form

$$T = \left[\lambda_1 \hat{p}_1 \,\middle|\, \lambda_2 \hat{p}_2 \,\middle|\, \lambda_3 \hat{p}_3 \right] \qquad \lambda_i \neq 0.$$

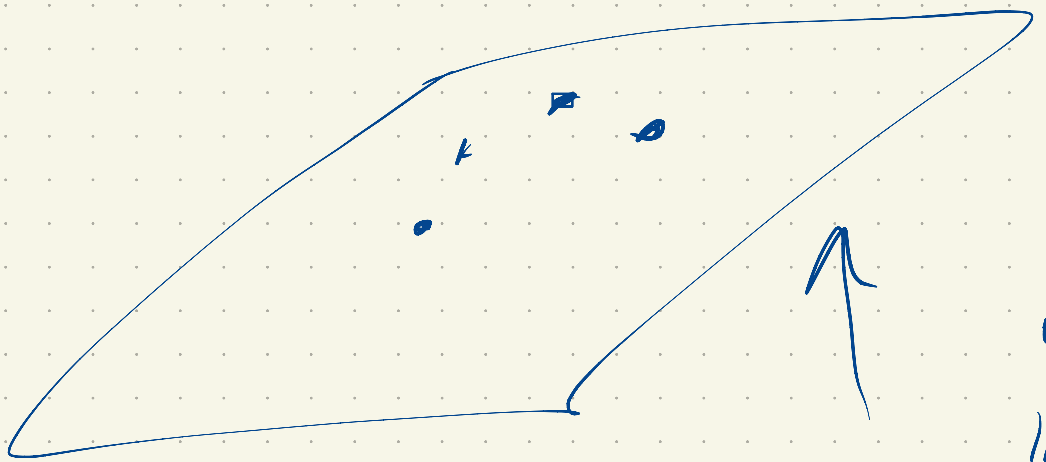
Then $T\hat{v}_4 = \lambda_1 \hat{p}_1 + \lambda_2 \hat{p}_2 + \lambda_3 \hat{p}_3$.

We want this to equal $\lambda_4 \hat{p}_4$ for some $\lambda_4 \neq 0$.

Let's try with $\lambda_4 = 1$. We want to solve

$$\left[\hat{p}_1 \,\middle|\, \hat{p}_2 \,\middle|\, \hat{p}_3 \right] \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \hat{p}_4$$

Claim: $\hat{P}_1, \hat{P}_2, \hat{P}_3$ are linearly independent. Indeed, if they were lin. dependent there is a plane thru O containing all three, in which case P_1, P_2, P_3 lie on a common projective line.



So: there exists a unique solution $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_3 \end{bmatrix}$.

We are done so long as we can show each $\lambda_i \neq 0$.

$$\lambda_1 \hat{P}_1 + \lambda_2 \hat{P}_2 = \hat{P}_4$$

This is ruled out because $P_1, \dots, P_4$ are in general position.