Projective Transformations

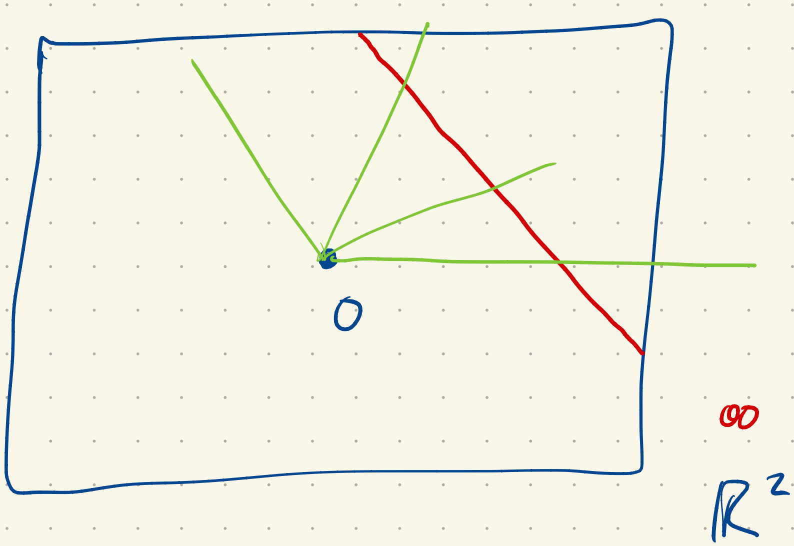
$\mathbb{R}P^2 \to \mathbb{R}P^2$

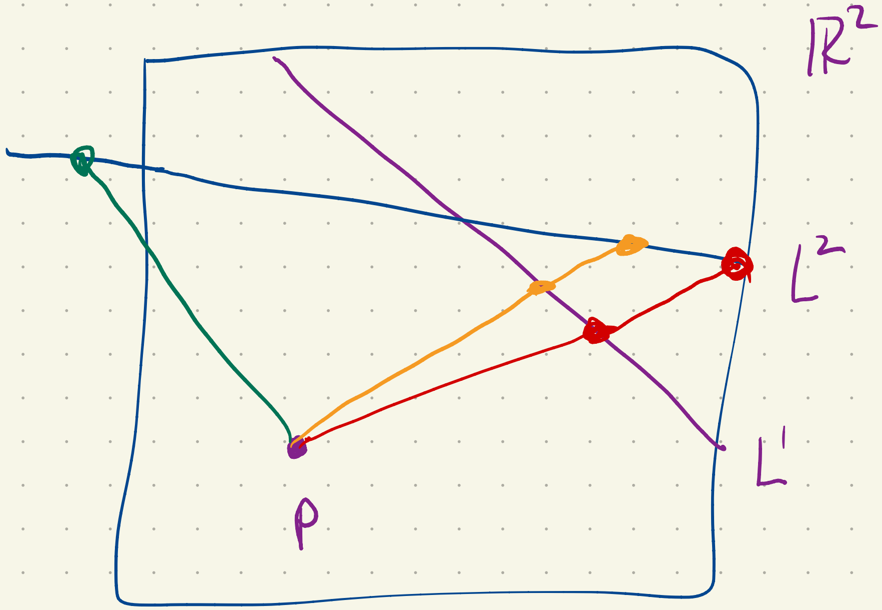
lines thru origin w/ 0 removed.

in $\mathbb{R}^3$

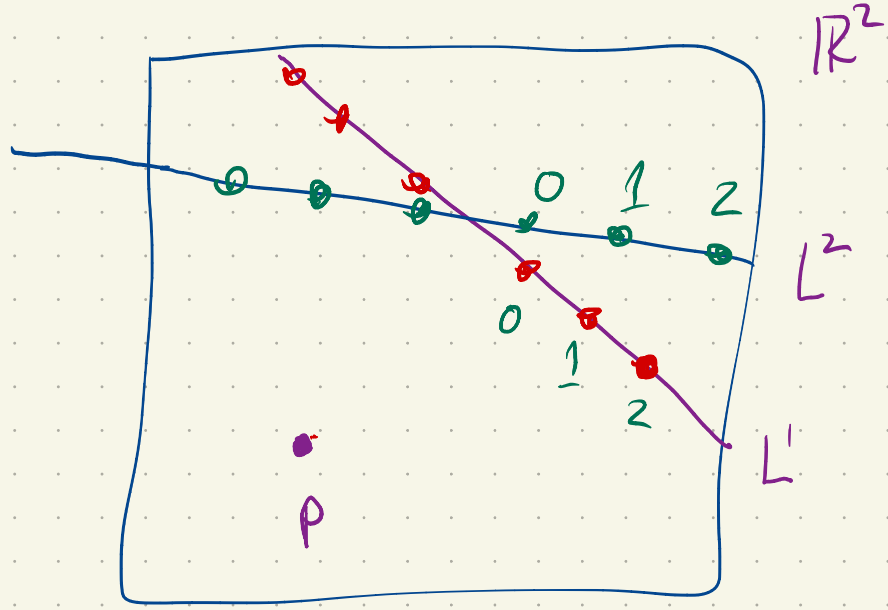
Projective line $\to$ lines thru origin in $\mathbb{R}^2$ w/ origin removed.

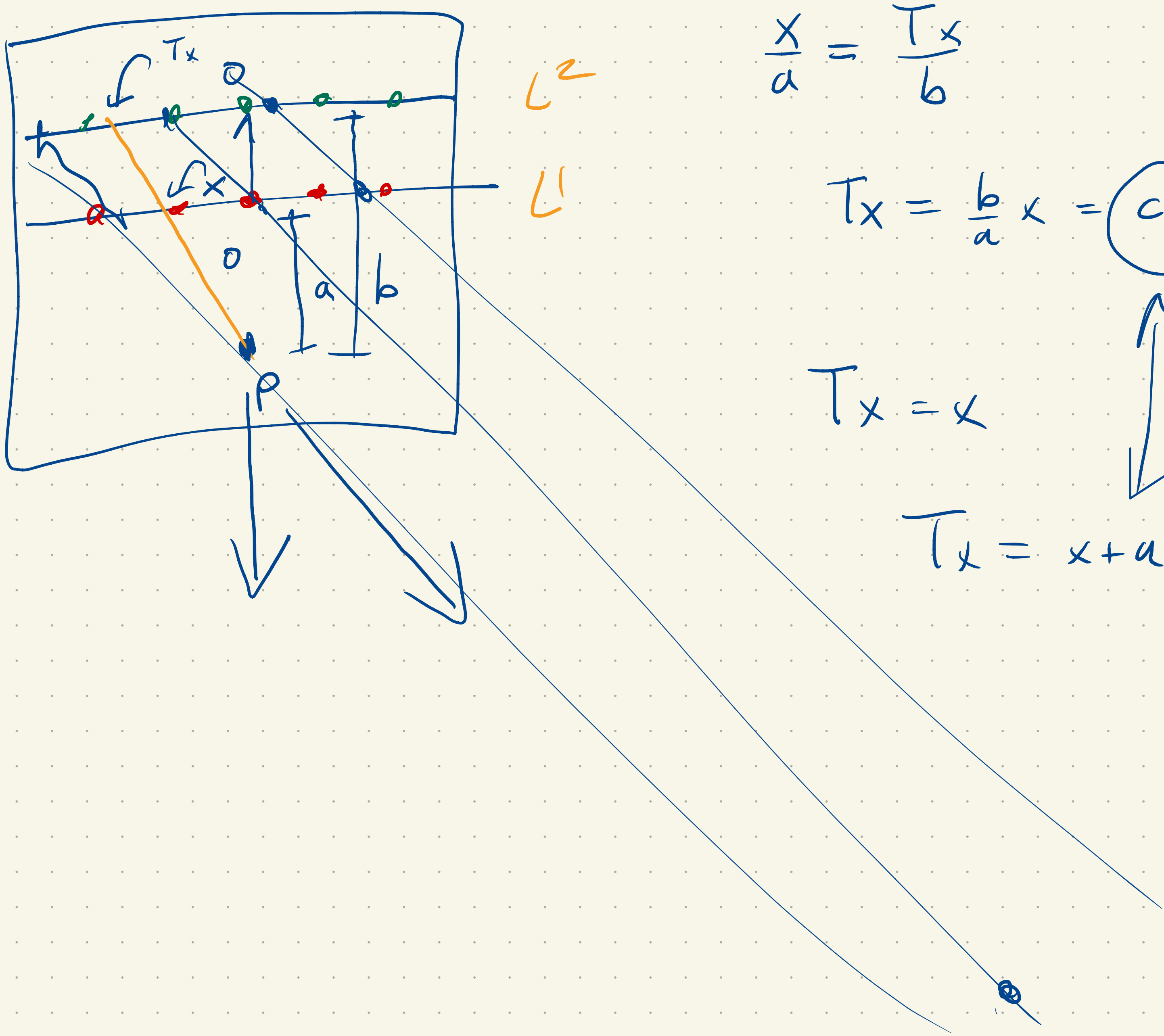
$\mathbb{R}P^1$





The projection through p of L^1 onto L^2.



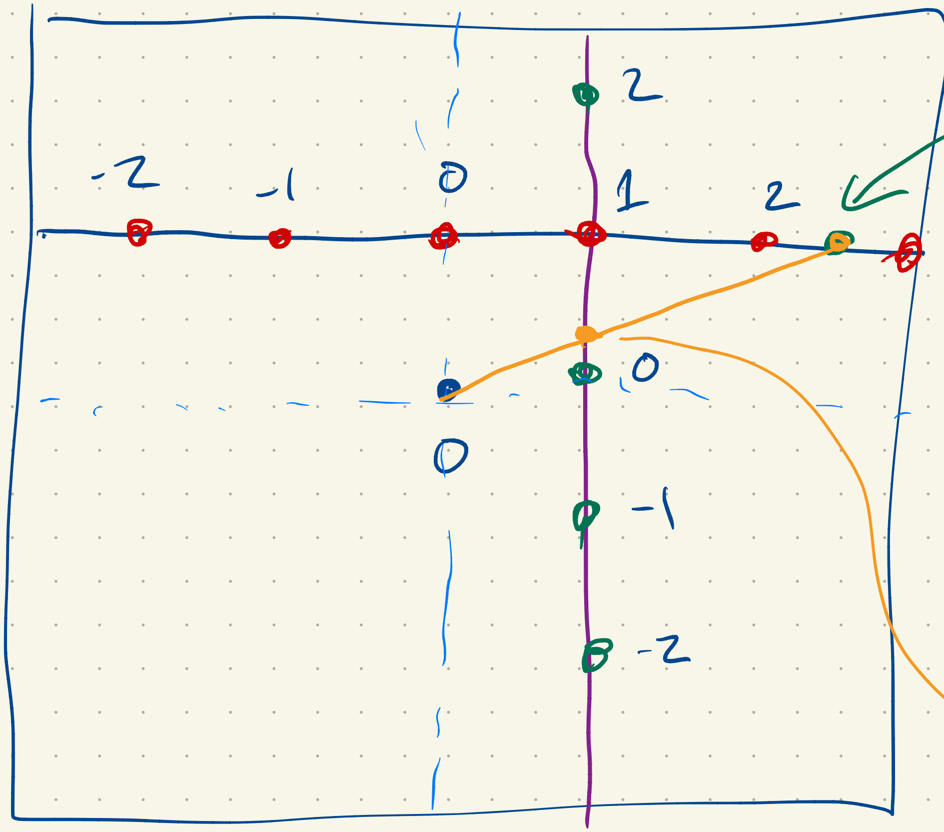


$$\frac{x}{a} = \frac{T_x}{b}$$

$$T_x = \frac{b}{a} x = cx$$

$$T_x = x$$

$$T_x = x + a$$



$(x, 1)$

L^1 $\quad y=1$

$1 \to 1$

$2 \to \frac{1}{2}$

$(1, y)$ $\quad (1, \frac{1}{x})$

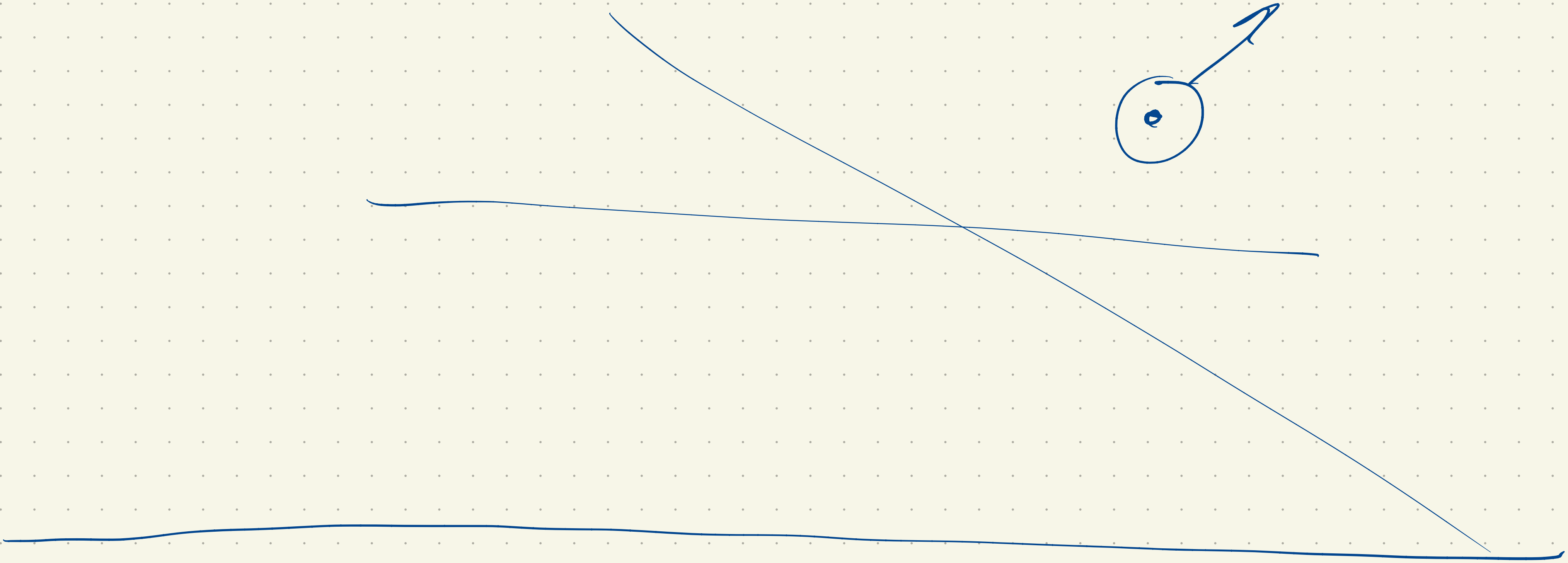
L^2 $\quad x=1$

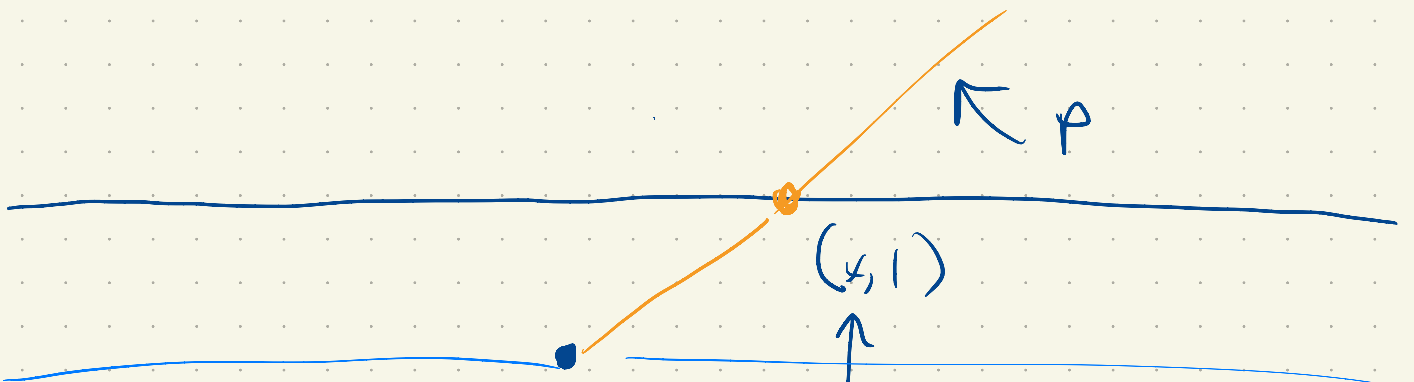
$Tx = cx$ $\quad c \in \mathbb{R}$

$Tx = x+a$ $\quad a \in \mathbb{R}$

$Tx = 1/x$

$$\frac{ax+b}{cx+d} \qquad ad-bc \neq 0 \qquad a,b,c,d \in \mathbb{R}$$





inhomogeneous coordinates of p

$(x, 1)$

$(\lambda x, \lambda)$

∞

Transformations should take lines thru origin to lines thru origin

(projective points to projective points)

Line to lines

linear maps

invertible!

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{cases} T(x+y) = T(x) + T(y) & x, y \in \mathbb{R}^2 \\ T(\lambda x) = \lambda T(x) & \lambda \in \mathbb{R} \end{cases}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = a e_1 + c e_2$

$T(e_2) = b e_1 + d e_2$

Tv $\qquad v = xe_1 + ye_2$

$$
\begin{aligned}
Tv &= T(xe_1 + ye_2) \\
&= T(xe_1) + T(ye_2) \\
&= x\,T(e_1) + y\,T(e_2) \\
&= x(ae_1 + ce_2) + y(be_1 + de_2)
\end{aligned}
$$

linearity

Matrix of T w.r.t the basis e_1, e_2

$$= (ax + by)e_1 + (cx + dy)e_2$$

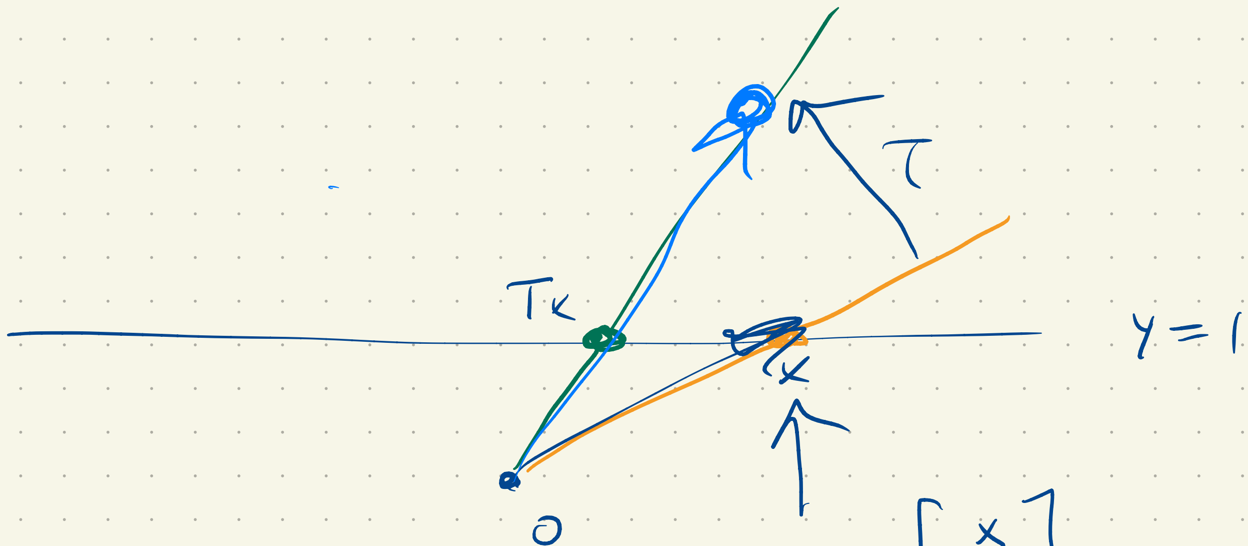
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

matrix we needed

$\hookrightarrow$ rep. of v w.r.t. e_1, e_2

$ad - bc \neq 0$

T what does T do to the point labeled by x?



$$\begin{bmatrix} x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \to \begin{bmatrix} x_0/y_0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax+b \\ cx+d \end{bmatrix} \to \begin{bmatrix} \frac{ax+b}{cx+d} \\ 1 \end{bmatrix}$$

$a, b, c, d \in \mathbb{R}$

$ad - bc \neq 0$

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ all act like the identity.

$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ λI

Transformation group: $\underbrace{GL(2,\mathbb{R})}_{}/\sim$

General linear group

2x2 invertible matrices

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$

$\lambda \neq 0$

$PGL(\mathbb{R}^2)$

projective general linear group.

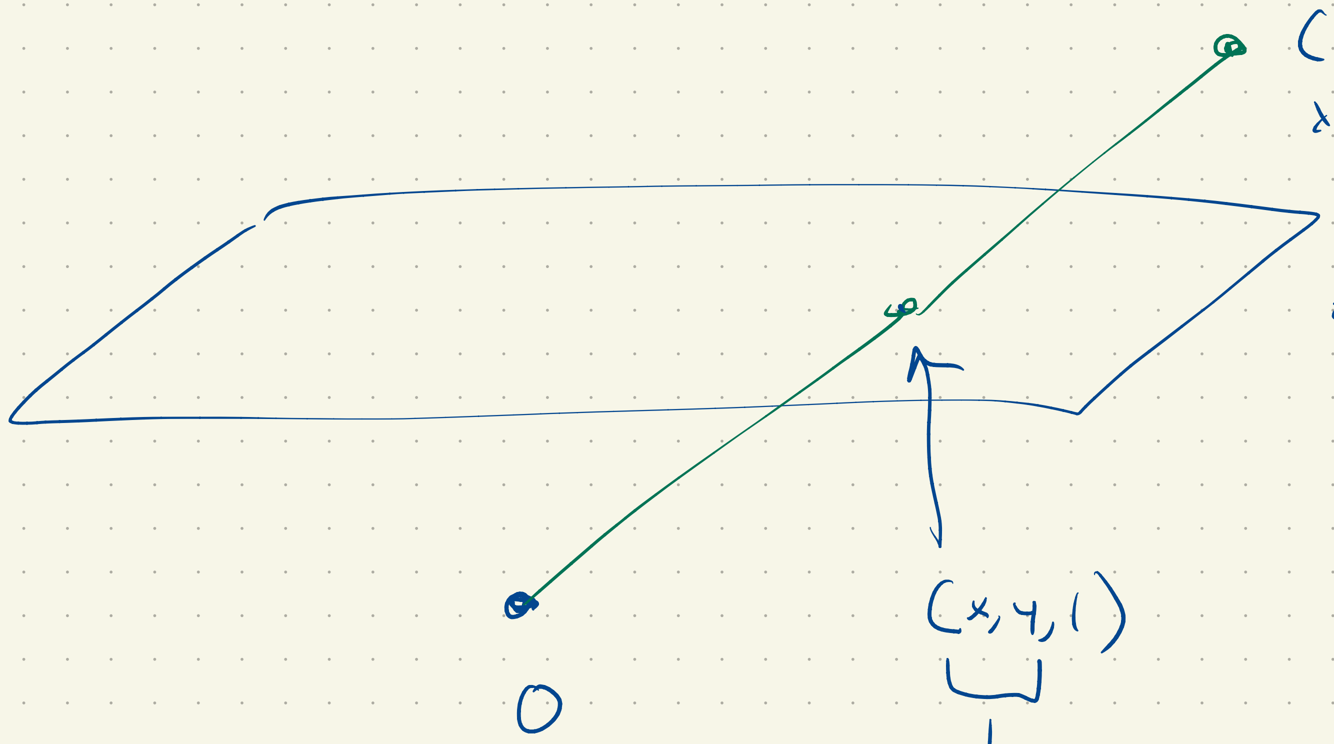
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftarrow T$$

What does T do to the projective point with inhomogeneous coord x?

$$\frac{ax+b}{cx+d}$$

$$(\lambda_1)\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\lambda_2)\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1\lambda_2\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

homogeneous coordinates of the result.



(x_0, y_0, z_0)

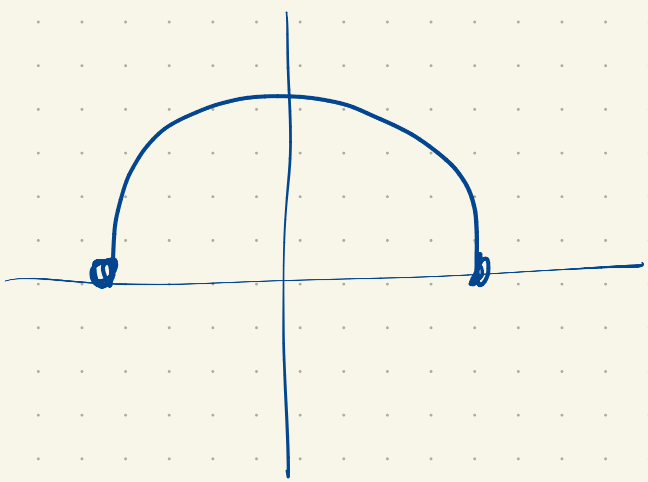
$z=1$

$\underbrace{(x, y}, 1)$

$\downarrow$

(x,y) label for a point in the projective plane.

$\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}, 1\right)$



Projective transformations will be liner maps e_1, e_2, e_3

d

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$
\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
=
\begin{bmatrix} ax + by + c \\ dx + ey + f \\ gx + hy + i \end{bmatrix}
$$

$$
\downarrow
$$

$$
\begin{bmatrix} \dfrac{ax + by + c}{gx + hy + i} \\[2ex] \dfrac{dx + ey + f}{gx + hy + i} \\[2ex] 1 \end{bmatrix}
$$

$$
(x, y) \longmapsto \left(\frac{ax + by + c}{gx + hy + i}, \quad \right)
$$

$$GL(3,\mathbb{R})/\sim \qquad A \sim \lambda A$$

$$Z \to GL(3,\mathbb{R})/\sim \qquad (\lambda I)A$$

$$PGL(\mathbb{R}^3) \to GL(3,\mathbb{R})/\sim$$

$$Z \to \{\lambda I : \lambda \neq 0\}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} ax + by + cz \\ \vdots \\ gx + hy + iz \end{bmatrix}$$

homog coords of point

homogeneous coords of the image