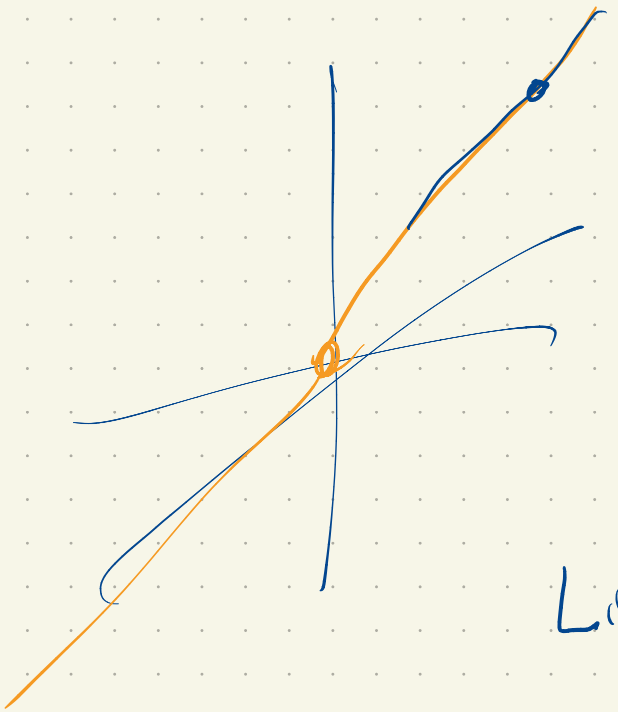
P_2: equivalence classes of points $p \in \mathbb{R}^3 \setminus \{0\}$ where

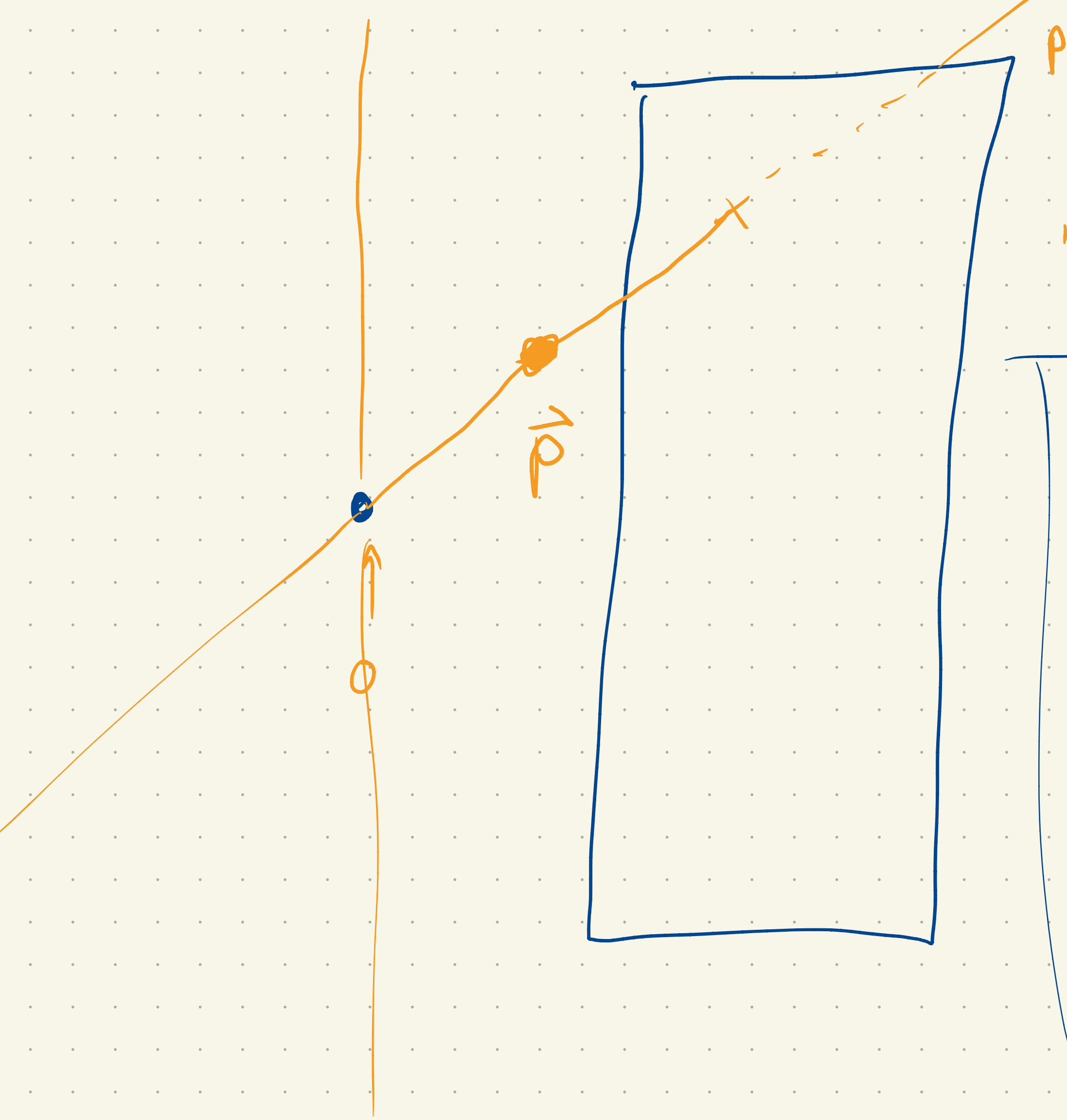
$$p \sim q \iff p = \lambda q \quad \lambda \neq 0$$



$p \in P_2$

$\vec{p} \in \mathbb{R}^3$

Lines: planes through the origin with 0 removed



"points at ∞"

Given a point $\vec{p}$ on $p \in P_2$

$$\vec{p} = \underbrace{(x, y, z)} \quad \text{not all coords are zero.}$$

$\downarrow$

the homogeneous coords of p

We call $(\lambda x, \lambda y, \lambda z)$ the homogeneous coords of p as well $(\lambda \neq 0)$

$(1,3,7) \qquad (2,9,3)$

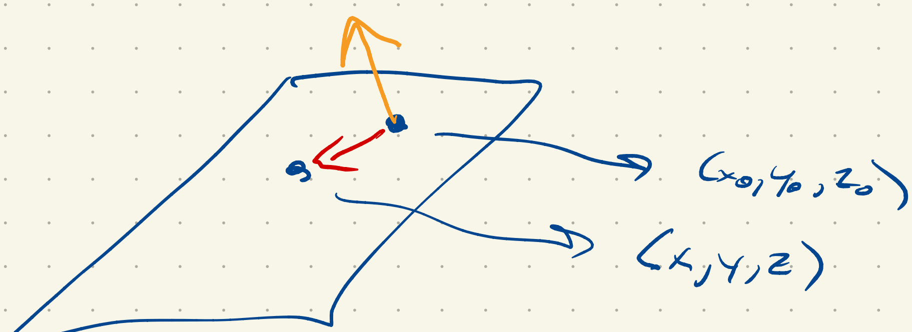
$\underbrace{\text{Projective lines}}$ have coordinates also!

$\rightarrow$ plane through the origin (+ legalese)

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$\langle A,B,C\rangle \longrightarrow$

$(x_0,y_0,z_0) \longrightarrow$ a point on the plane. (you have choices



$$Ax + By + Cz = 0$$

$$\boxed{Ax + By + Cz = D}$$

$[A, B, C]$ are the homogeneous coordinates of the line.
↑
projective

$[\lambda A, \lambda B, \lambda C]$ are homogenous coords of the same ~~plane~~ line $(\lambda \neq 0)$

$\mathcal{L}$ projective line | $L \to [A, B, C]$ homog. coords.

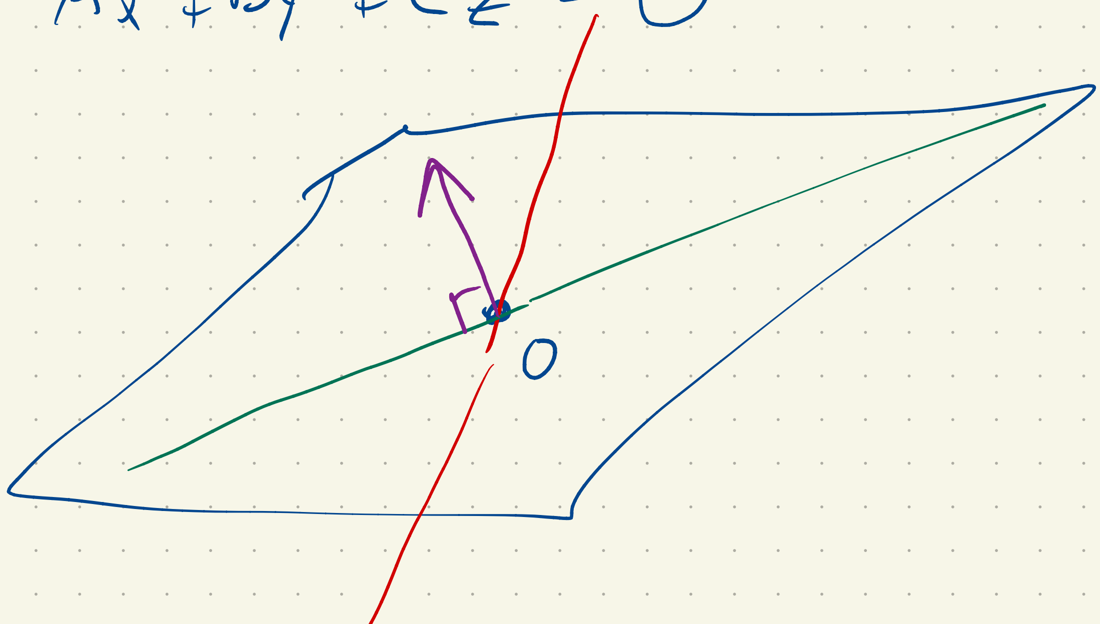
p - - - point | $(x, y, z) \to \vec{p}$ homog. coords

Given $\mathcal{L}$ and p how can we tell, via homogeneous coordinates, that p is on $\mathcal{L}$ (or $\mathcal{L}$ is on p)

$L = [A, B, C]$ $\qquad$ $\vec{p} = (x, y, z)$

$[\lambda A, \lambda B, \lambda C]$ $\qquad$ $(\lambda x, \lambda y, \lambda z)$

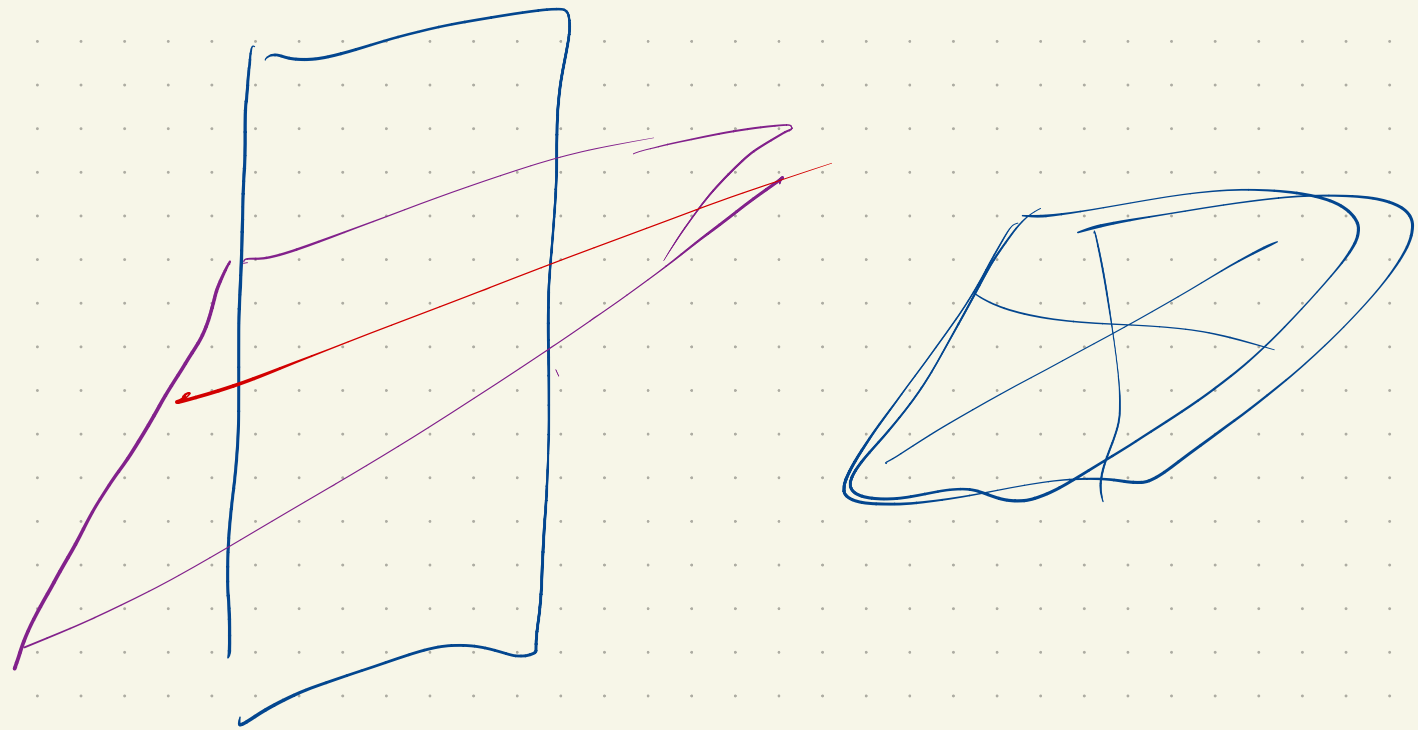
$Ax + By + Cz = 0 \iff$ incidence.



For the projective plane we have a phenomenon known as duality. True statements involving points and lines remain true interchanging lines and points.

First example: Given two distinct projective points there is a unique projective line incident to both.

Dual statement: Given two distinct projective lines there is a unique projective point incident to both.



Pf: Let p_1, p_2 be distinct points in P_2 with homogeneous coordinates $\vec{p}_1$ and $\vec{p}_2$.

Let $(A, B, C) = \vec{p}_1 \times \vec{p}_2$.

Recall that $\vec{a} \times \vec{b} = \vec{0}$ iff $\vec{a}$ and $\vec{b}$ are colinear.

So $(A,B,C) \neq 0$ and $[A,B,C] = L$ are

homogeneous coords for a line. By properties of the
$\mathcal{L}$.

cross product $(A,B,C) \perp \vec{p}_i$ $i = 1, 2$.

So if $\vec{p}_i = (x_i, y_i, z_i)$, $Ax_1 + Bx_2 + Cx_3 = 0$

and $\vec{p}_1$ is on $\mathcal{L}$. Similarly $\vec{p}_2$ is on $\mathcal{L}$.

To establish uniqueness let $\hat{L}$ be homogeneous

coords of a line incident with p_1, p_2.

Then $\hat{L} \cdot \vec{p}_1 = 0$ and $\hat{L} \cdot \vec{p}_2 = 0$.

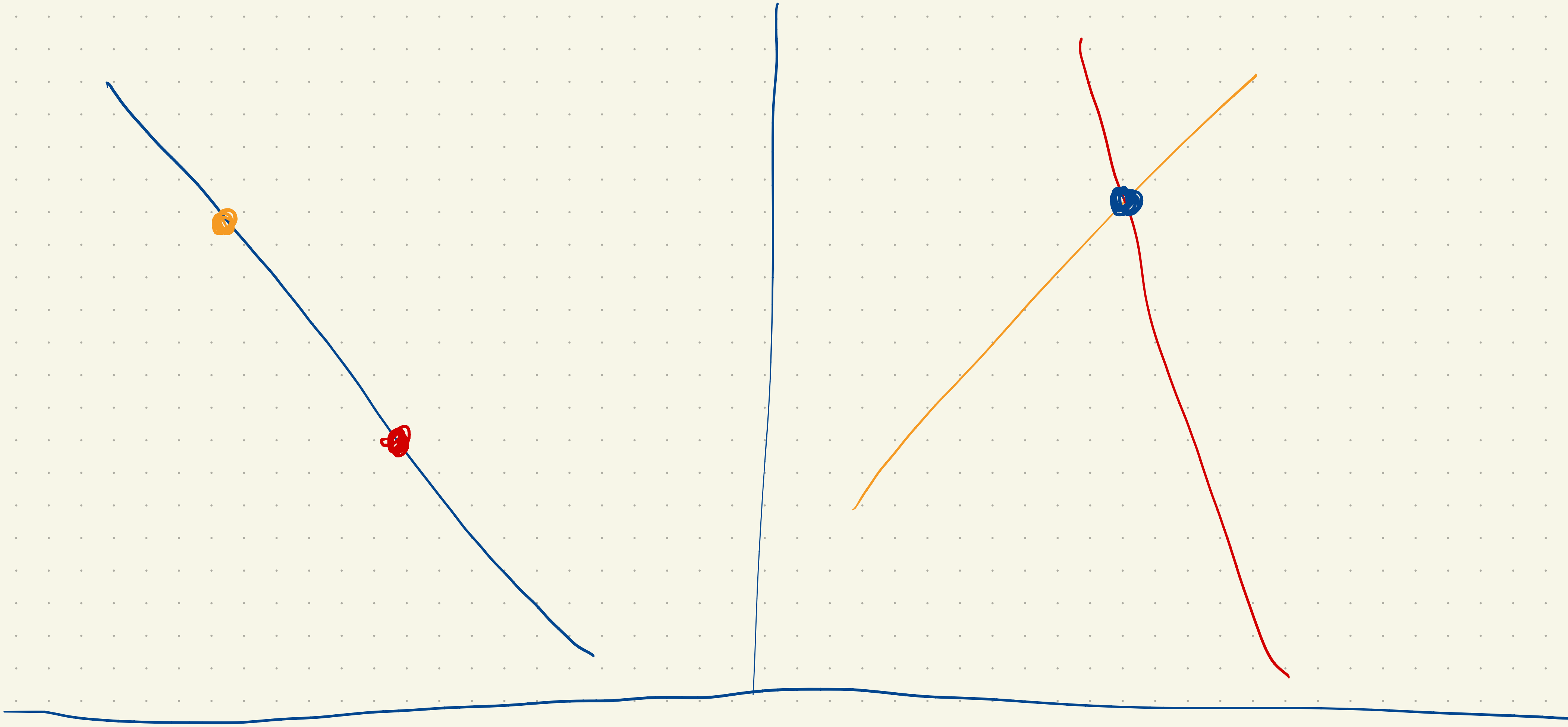
Now from vector calculus

$$\hat{L} \times \underbrace{(\vec{p}_1 \times \vec{p}_2)}_{L} = \vec{p}_1 \underbrace{(\hat{L} \cdot \vec{p}_2)}_{0} - \vec{p}_2 \underbrace{(\hat{L} \cdot \hat{p}_1)}_{0}$$

$$= 0 - 0$$

$$= 0.$$

So $\hat{L}$ and L are colinear and $\hat{L} = \lambda L$ for some $\lambda \neq 0$. This establishes uniqueness.



$$(x, y, z)$$

$$\updownarrow$$

$$[x, y, z]$$

$\vec{0}$

0

I