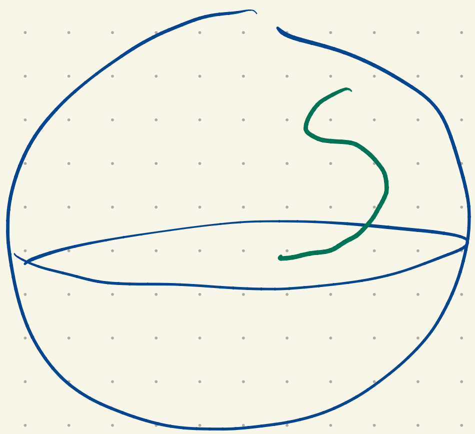
Arclength We inherit it from the sphere

S^2



$$\gamma(t) = (x(t), y(t), z(t))$$

$$\int_a^b |\dot{\gamma}(t)|\,dt$$

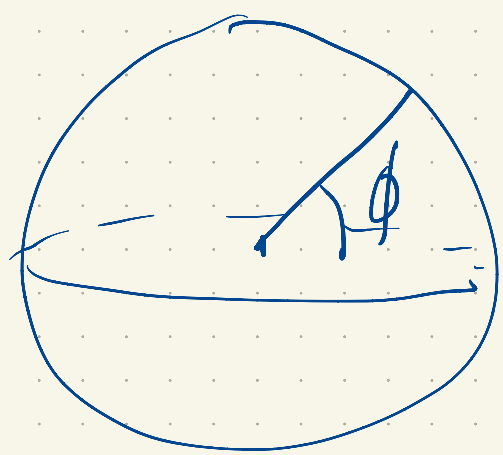
$$\downarrow$$

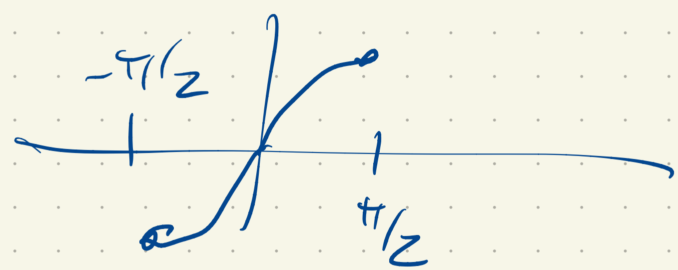
$$\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

in $\mathbb{R}^3$

θ, ϕ

$x = \cos\theta\cos\phi$, , $z = \sin\phi$



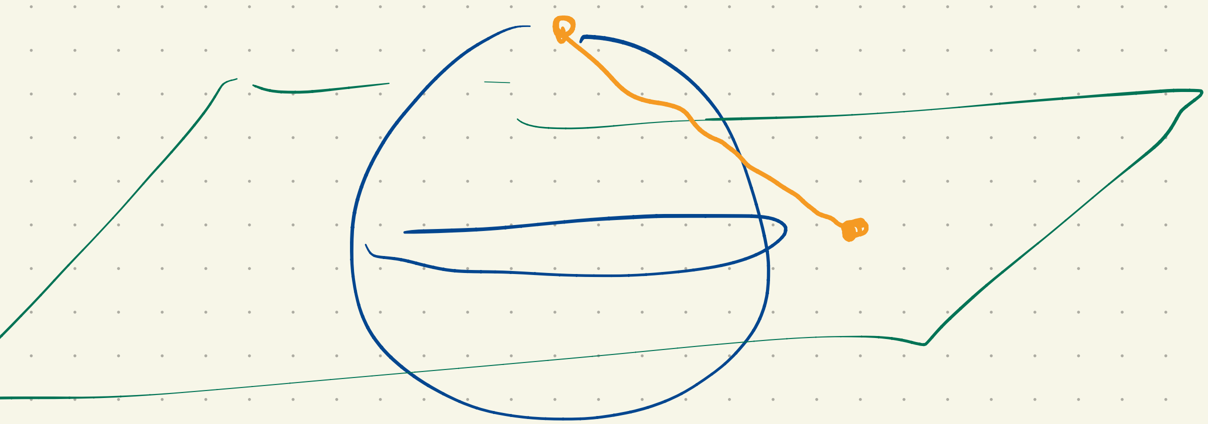


$$\int_a^b \sqrt{\dot{\phi}^2 + \cos^2\phi\, \dot{\theta}^2}\; dt$$

$$z = \sin\phi$$

$$\dot{z} = \cos\phi\, \dot{\phi}.$$

$$\dot{z}^2 = \cos^2\phi\, \dot{\phi}^2$$



$$(r, \underline{\vartheta})$$

$$\theta = \vartheta$$

$$x^2+y^2+z^2=1$$

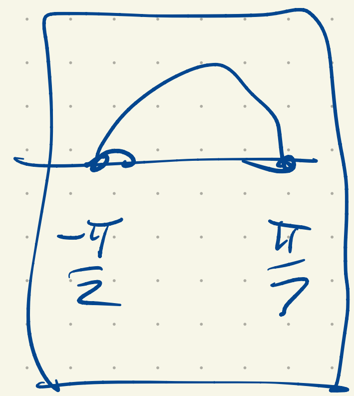
$$(x,y,z) \longmapsto \frac{x+iy}{1-z}$$

$$r = \left| \frac{x+iy}{1-z} \right| = \frac{\sqrt{x^2+y^2}}{|1-z|} = \frac{\sqrt{1-z^2}}{1-z}$$

$$\sqrt{\cos^2\phi}$$

$$= \frac{\sqrt{1-\sin^2\phi}}{1-\sin\phi}$$

$$= \frac{\cos\phi}{1-\sin\phi}$$



$$r = \frac{\cos\phi}{1-\sin\phi}$$

$$r^2 = \frac{1+\sin\phi}{1-\sin\phi} \qquad \sin\phi = \frac{r^2-1}{r^2+1} \qquad 1-\sin\phi = \frac{2}{1+r^2}$$

$$\int_0^b \sqrt{\dot\phi^2 + \cos^2\phi\,\dot\theta^2}\,dt$$

rewrite in terms of

r, Θ

$$\dot\theta = \dot\Theta \qquad \theta = \Theta$$

$$r = \frac{\cos\phi}{1-\sin\phi} \Rightarrow \dot r = \frac{1}{1-\sin\phi}\dot\phi$$

$$(1-\sin\phi)\,\dot r = \dot\phi$$

$$\dot\phi^2 + \cos^2\dot\theta^2 = (1-\sin\phi)^2\,\dot r^2 + \cos^2\phi\,\dot\Theta^2$$

$$= (1-\sin\phi)^2 \left[\dot{r}^2 + \left(\frac{\cos\phi}{1-\sin\phi}\right)^2 \dot{\theta}^2 \right]$$

$$= \left(\frac{2}{1+r^2}\right)^2 \left[\dot{r}^2 + r^2 \dot{\theta}^2 \right]$$

$$\sqrt{\dot{\phi}^2 + \cos^2\phi\, \dot{\theta}^2} = \frac{2}{1+r^2} \boxed{\sqrt{\dot{r}^2 + r^2 \dot{\theta}^2}}$$

$$z = a + ib = r\cos\theta + i\, r\sin\theta$$

$$\dot{z} = \dot{a} + i\dot{b} = \dot{r}\left[\cos\theta + i\sin\theta\right] + r\left[-\sin\theta + i\cos\theta\right] \dot{\theta}\, i$$

$$|\dot{z}|^2 = \dot{r}^2 + r^2 \dot{\theta}^2$$

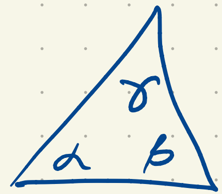
Arclength

$$\int_a^b \frac{2}{1+r^2} |\dot{z}| \, dt$$

Arclength is invariant.

Comparison

Comparison $I = \pi$



unit sphere

Hyperbolic

$$A = \pi - (\alpha + \beta + \gamma)$$

$$\int_0^b \frac{2}{1-r^2} |\dot{z}| \, dt$$

$$\iint_\Omega \frac{4r}{(1-r^2)^2} \, dr \, d\theta$$

Euclidean

$$\alpha + \beta + \gamma = \pi$$

$$\int_a^b |\dot{z}| \, dt$$

$$\iint_\Omega r \, dr \, d\theta$$

Elliptic

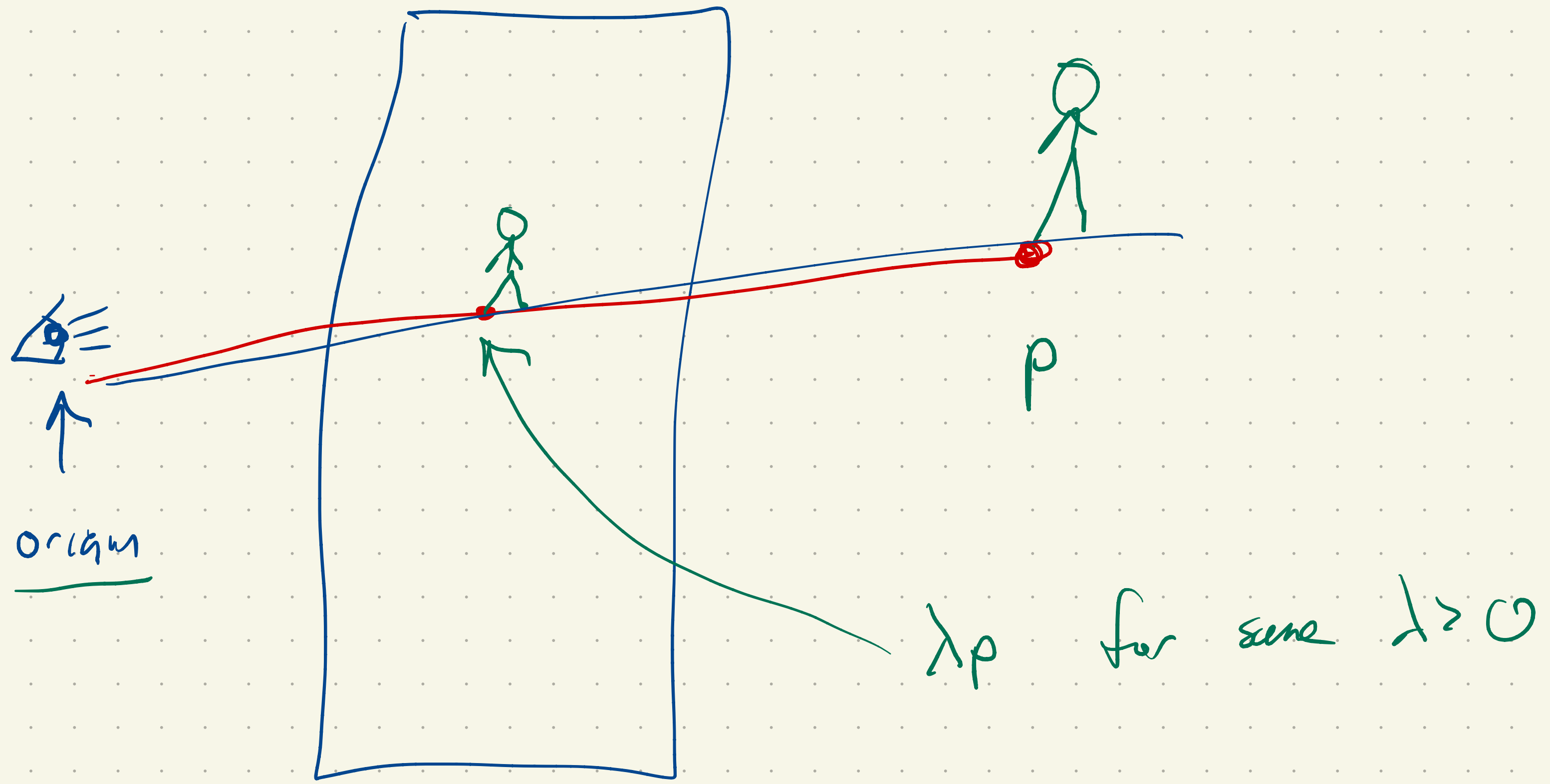
$$A = \pi + (\alpha + \beta + \gamma)$$

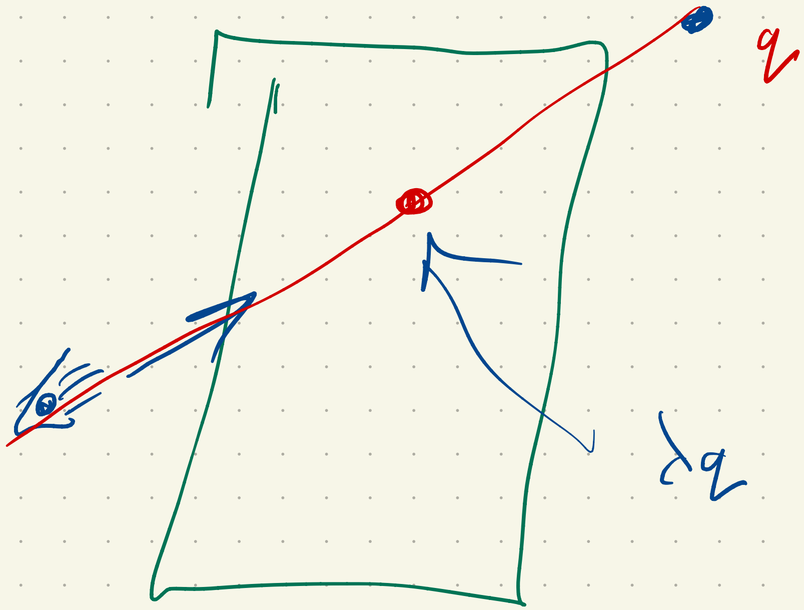
$$\int_a^b \frac{2}{1+r^2} |\dot{z}| \, dt$$

$$\iint_\Omega \frac{4}{(1+r^2)^2} r \, dr \, d\theta$$

Projective Geometry.

Based on monocular vision



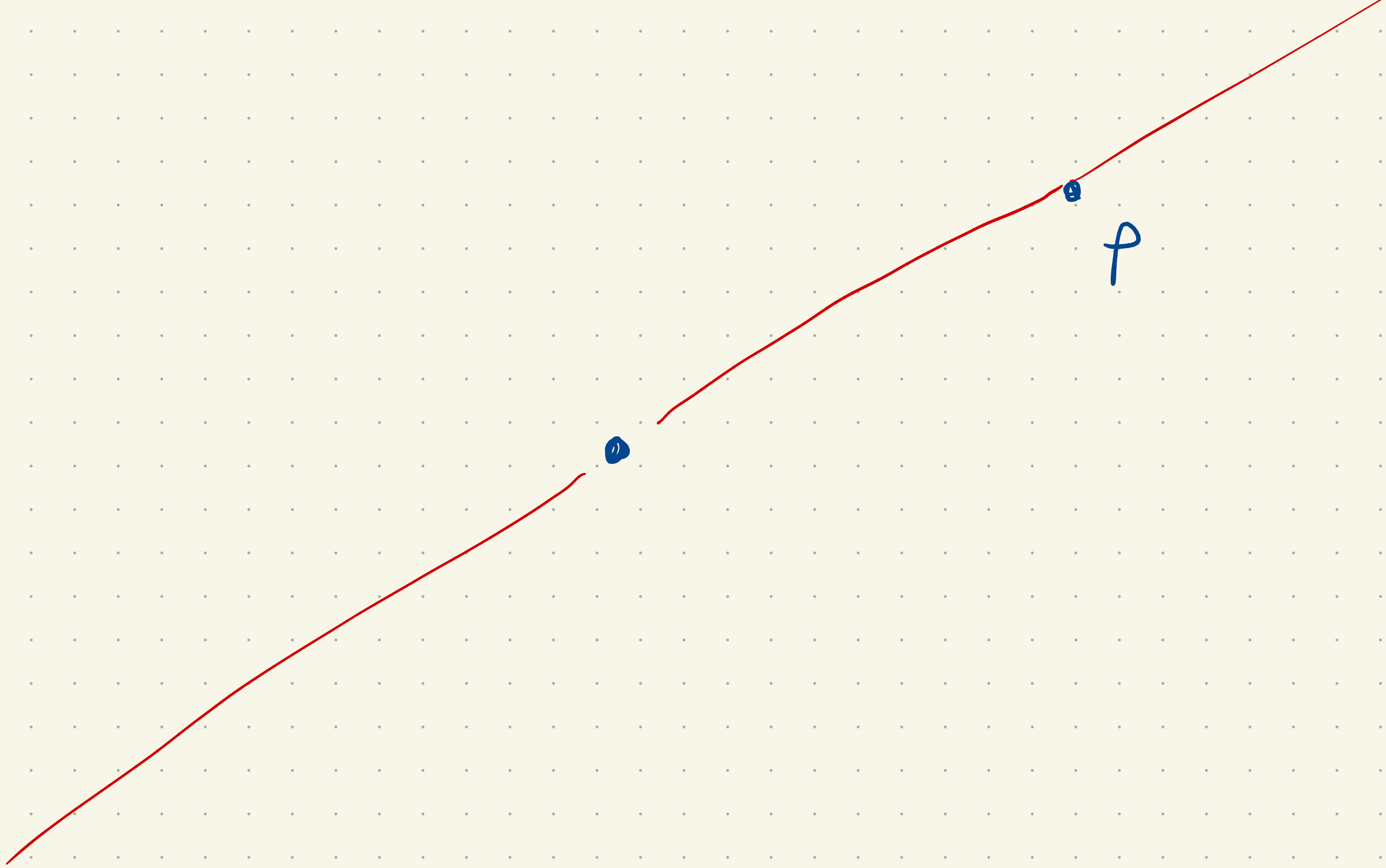


λq for some $\lambda \geq 0$.

We will identify all points in $\mathbb{R}^3 \setminus \{0\}$

where $p \sim \lambda p \quad \lambda > 0$.

Twist: We'll identify $p \sim \lambda p \quad \lambda \neq 0$

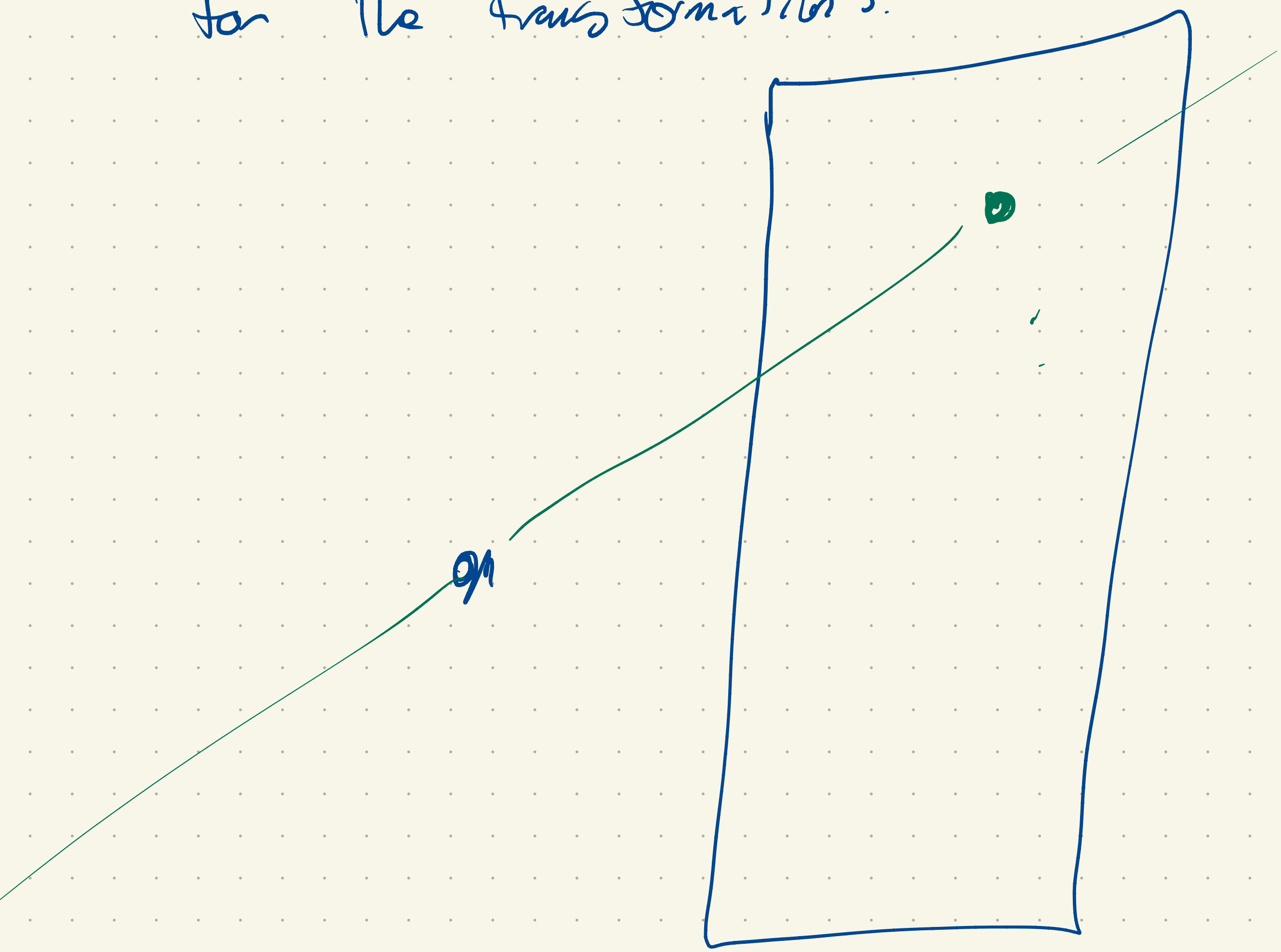


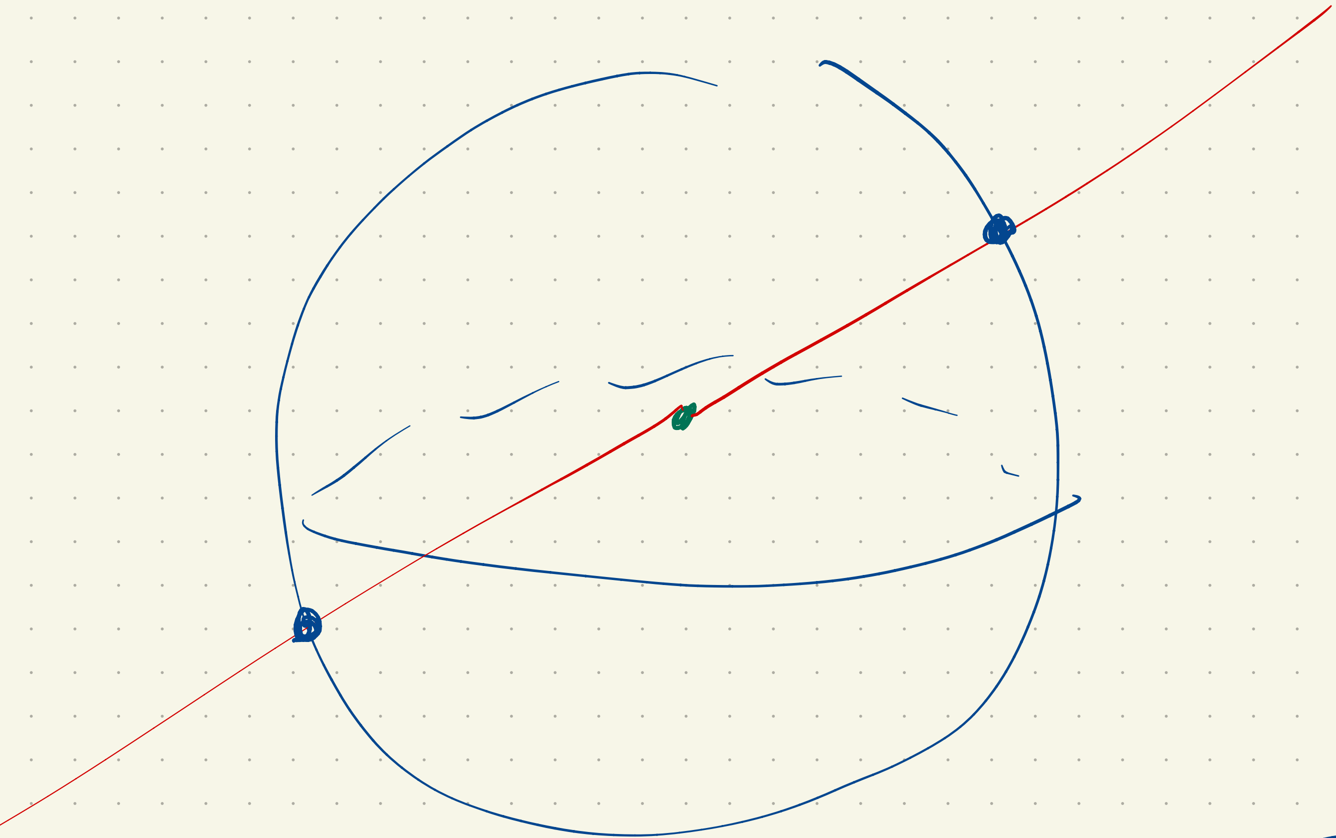
P^2 is the set of equivalence classes of points in $\mathbb{R}^3 \setminus \{0\}$ where

$$p \sim q \iff p = \lambda q \text{ for some } \lambda \neq 0.$$

These are thought of as lines through the origin in $\mathbb{R}^3$

These are the points in our new geometry. Stand by for the transformations.

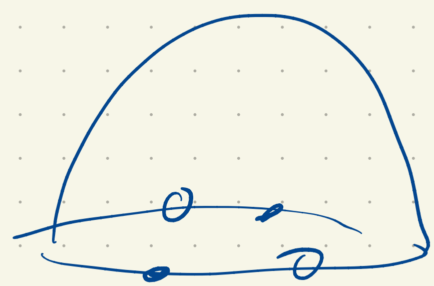




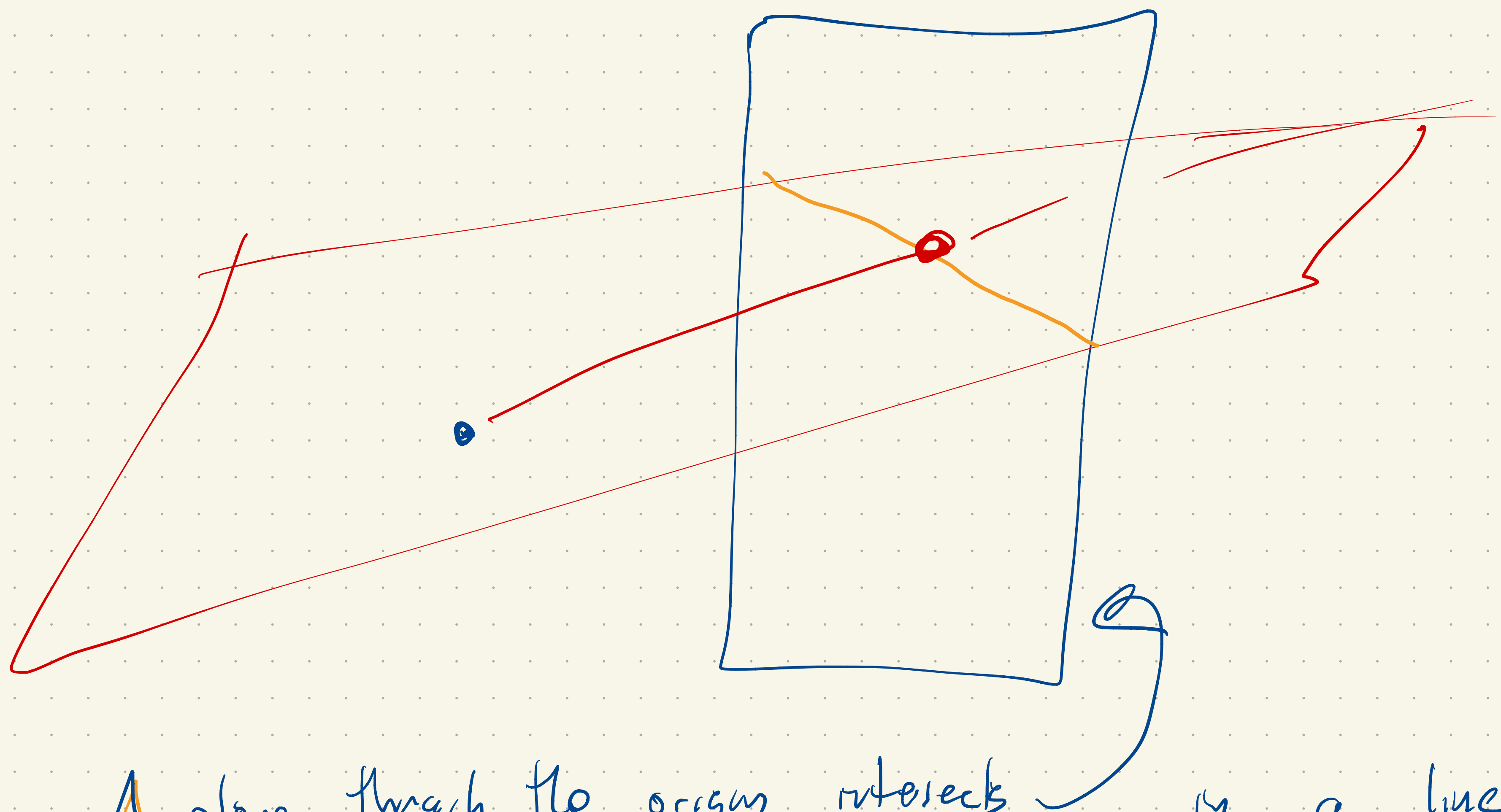
$$S^2/\sim$$

$$\uparrow$$

$$p \sim -p$$



The points of this geometry can be thought of as the points of single elliptic geometry.



A plane through the origin intersects in a line.

Def: A line in P^2 is a plane through the origin in $\mathbb{R}^3$ with the origin removed.