Elliptic Geometry

First version: geometry of the sphere

$a^2+b^2+c^2=1$

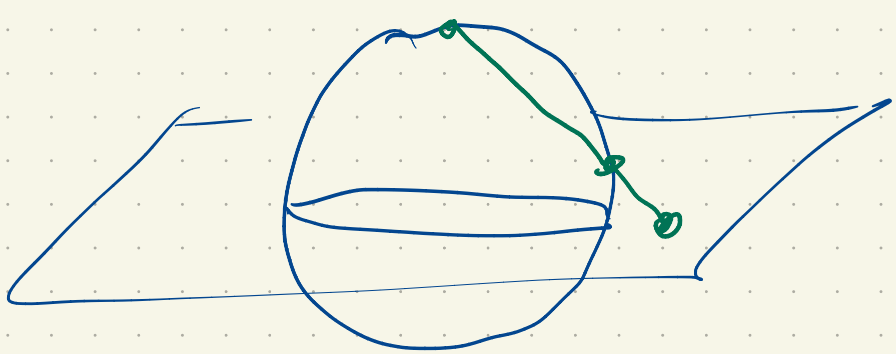
(a,b,c)

invariants: angle
- arclength
- area
- distance between points

Transformation group: rigid rotations of $\mathbb{R}^3$

3×3 matrices A (Those form a group!)

$$A^T A = I$$



$$\sigma(a,b,c) = \frac{a+ib}{1-c} \in \mathbb{C}$$

$$\sigma(0,0,1) = \infty$$

Recall: $p_1, p_2 \in S^2$ $\qquad z_i = \sigma(p_i)$

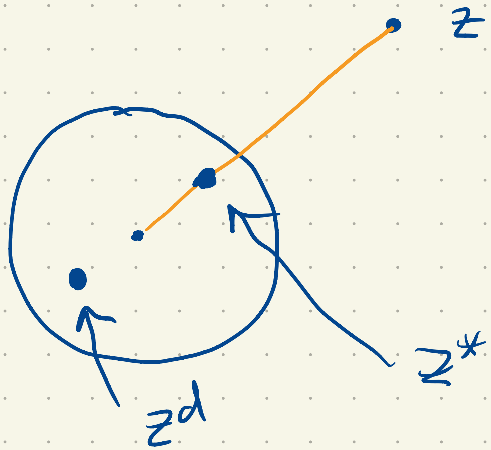
$p_1 = -p_2 \iff z_1 \bar{z}_2 = -1$ $\quad$ (HW #2, #3?)

Interested in Möbius transformations that preserve

$$\overline{z^d} = \frac{-1}{z}$$

$$z^d = \frac{-1}{\bar{z}} \qquad z\,\overline{z^d} = -1$$

We want $T(z^d) = T(z)^d$ | Compare for hyperbolic transformations $T(z^*) = T(z)^*$



$z^* \to$ reflection about S^1

$z^* = 1/\bar{z}$

$z^d = -z^*$

Who are the Möbius transformations satisfying $T(z^d) = T(z)^d$?

$$T(z) = \frac{az+b}{cz+d}$$

WLOG: $ad-bc=1$

almost determines a,b,c,d

$-a,-b,-c,-d$

$$T(z^d) = \frac{-\frac{a}{\bar z}+b}{-\frac{c}{\bar z}+d} = \frac{-a+b\bar z}{-c+d\bar z}$$

$$T(z)^d = -\frac{\bar c\bar z+\bar d}{\bar a\bar z+\bar b}$$

$$\frac{bw-a}{dw-c} = \frac{-\bar c w-\bar d}{\bar a w+\bar b} \quad \forall\, w\in\mathbb{C}$$

$-bc+ad=1 \qquad -\bar c\bar b+\bar a\bar d = \overline{ad-bc} = \bar 1 = 1$

$b = -\bar{c}, \quad a = \bar{d}$

$\boxed{b = \bar{c} \quad a = -\bar{d}}$

↳ Exercise: this is impossible

$$\boxed{Tz = \frac{az + b}{-\bar{b}z + \bar{a}}}$$

$a\bar{a} + b\bar{b} = 1 \qquad \boxed{|a|^2 + |b|^2 = 1}$

Elliptic transformation

$$= \frac{a}{\bar{a}} \frac{z + b/a}{-\bar{b}/\bar{a}\, z + 1}$$

$$p = -\frac{b}{a}$$

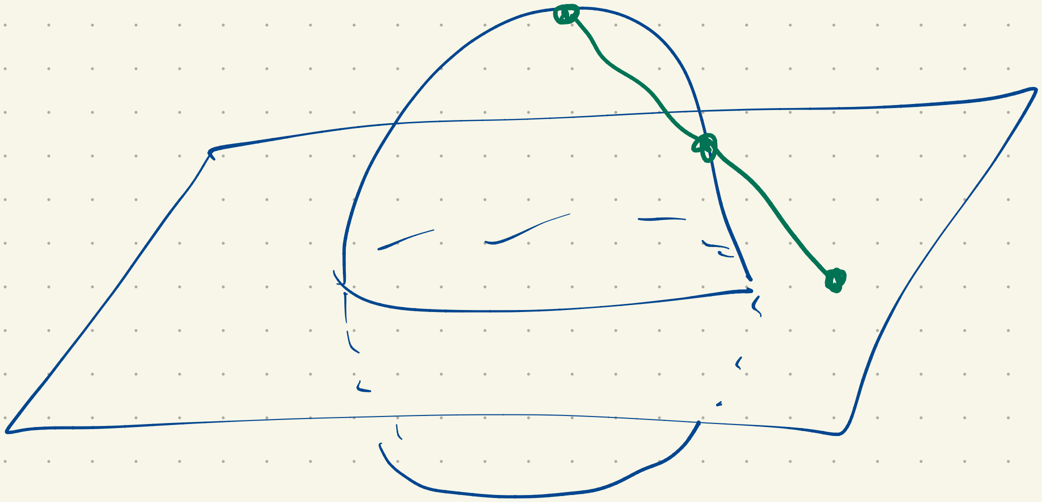
$$\bar{p} = -\bar{b}/\bar{a}$$

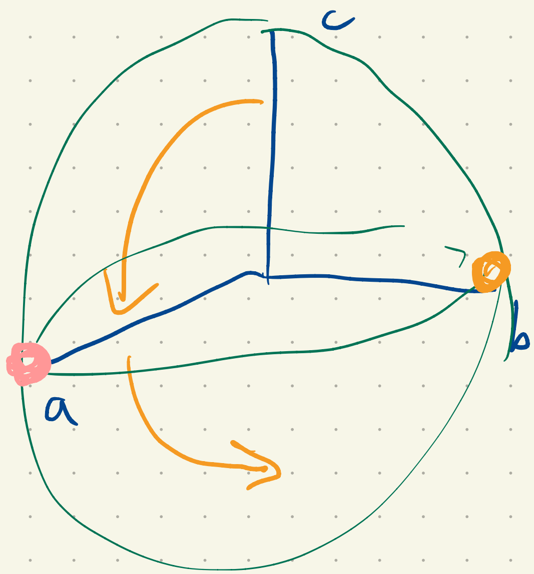
$$= e^{i\theta} \frac{z - p}{1 + z\bar{p}}$$

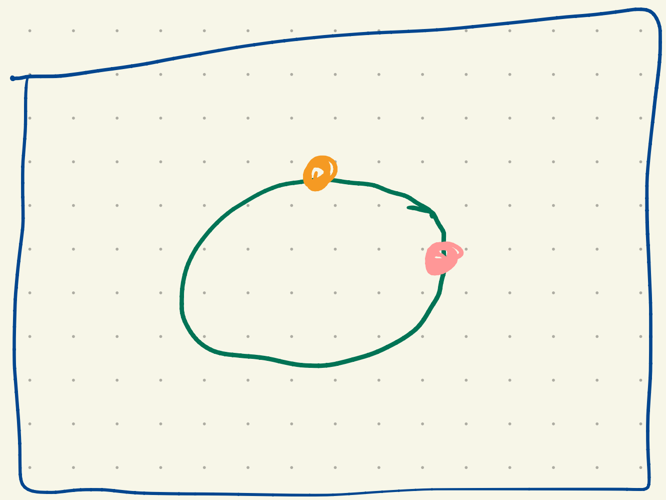
Compare: $e^{i\theta} \dfrac{z - p}{1 - z\bar{p}} \qquad |p| < 1$

for hyperbolic transf.

e.g. $T(z) = e^{i\theta} z$ $(p = 0)$







$\infty \to 1$

$1 \to 0$

$-1 \to \infty.$

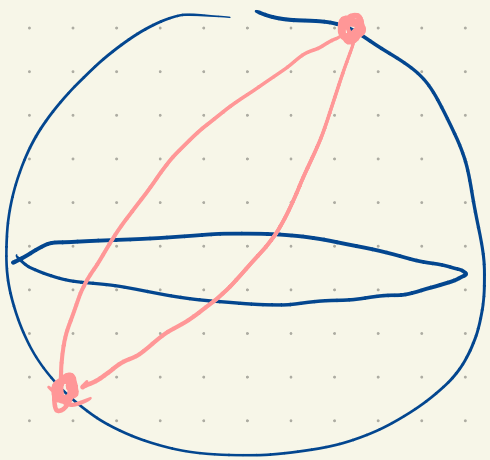
$$\frac{az+b}{-\bar{b}z+\bar{a}} \qquad |a|^2+|b|^2=1$$

$$\frac{z-1}{z+1}\cdot\frac{\infty+1}{\infty-1} = \frac{z-1}{z+1}$$

$a=1$

$b=-1$

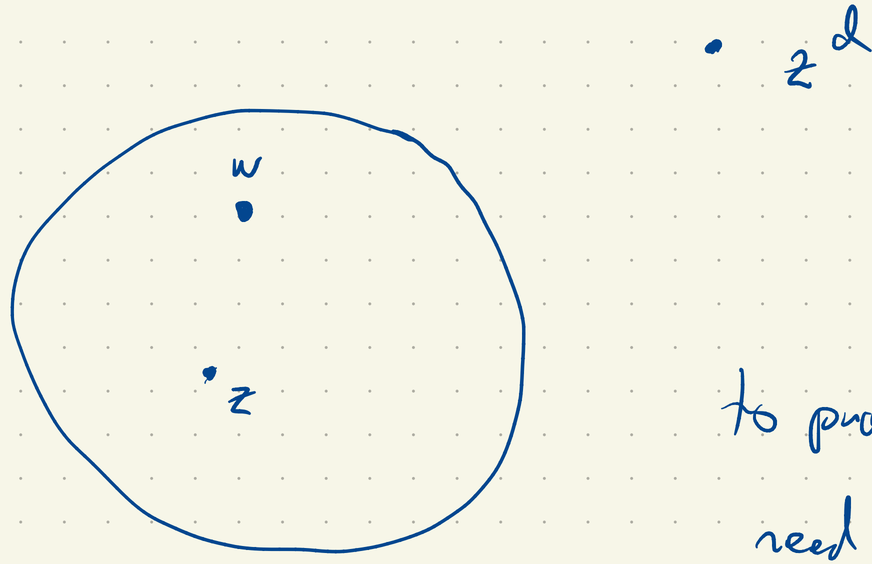
$|a|^2+|b|^2=1$



Def: An elliptic line is a Möbius line L such that $\boxed{\text{for all}}$ $z\in L$, $z^d \in L$.

In fact, we can replace the for all with a "there exists"

Proposition: A Möbius line L is an elliptic line iff there exists $z\in L$ such that $z^d\in L$.



w, z, z^d all on one Möbius line.

to prove the proposition one would need to show w^d is on the same line

$(p_0, p_1, p_2, p_3) \in \mathbb{R}$ iff on same Möbius line.

On your HW: z, z^d, w $\quad w \neq z, w \neq z^d$

$$(w, w^d, z, z^d) \in \mathbb{R}$$

Cor: Elliptic transformations take elliptic lines to elliptic lines.

$L \leftarrow$ elliptic line

$T \leftarrow$ elliptic transf.

$\underbrace{T(L)}$ = elliptic line ?

a) Möbius line

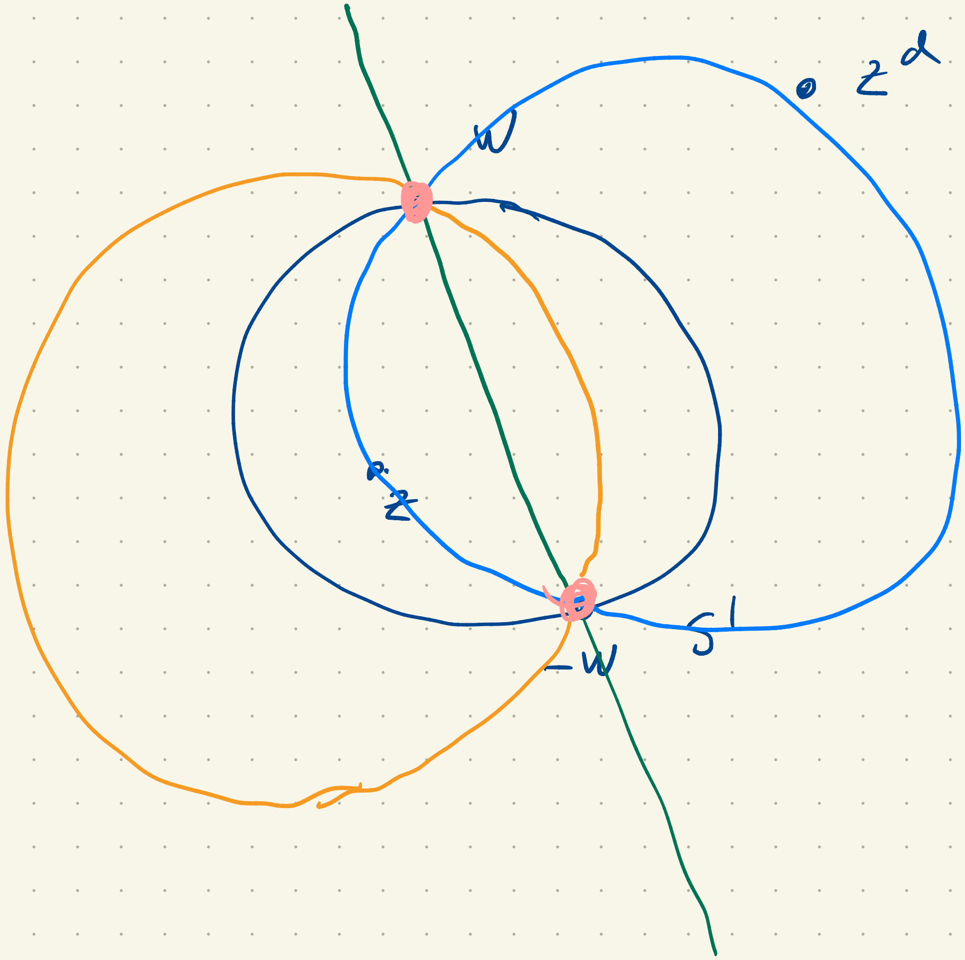
b) contain an antipodal pair

$z, z^d \in L$

$Tz \in T(L)$

$\underbrace{T(z^d)} \in T(L)$

$T(z^d) = (T(z))^d$



$$w^d = -\frac{1}{\bar{w}} = -w$$

$$w\bar{w} = |w|^2 = 1$$

$$\bar{w} = w^{-1}$$

$$\bar{w} = 1/w$$