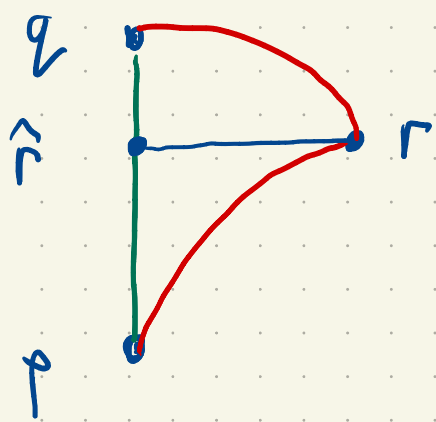
Properties of distance

- $d_H(p,q) \geq 0 \quad (=0 \iff p=q)$

  $$\left|\ln\left((p,q,r_1,r_2)\right)\right|$$

- $d_H(p,q) = d_H(q,p)$

- $\underbrace{d_H(p,q) \leq d_H(p,r) + d_H(r,q)}_{\text{triangle inequality}}$



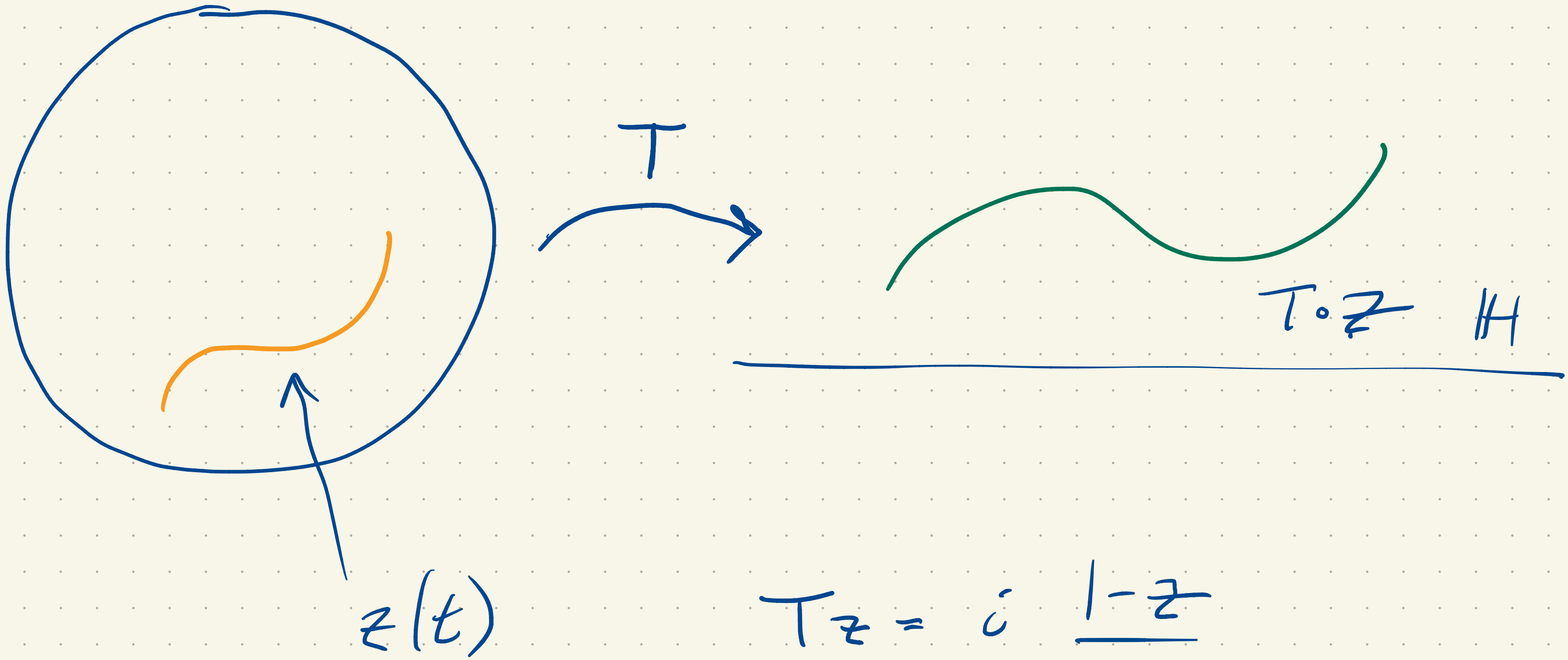
$$d_H(p,q) = d_H(p,\hat{r}) + d_H(\hat{r},q)$$

$$\boxed{d_H(p,\hat{r}) \leq d_H(p,r)}$$

$$\boxed{d_H(\hat{r},q) \leq d_H(r,q)}$$

- If S is a hyperbolic transf.

$$d_H(Sp, Sq) = d_H(p,q)$$



$$Tz = i\,\frac{1-z}{1+z}$$

$$w = T(z(t))$$

$$w = i\,\frac{1-z}{1+z}$$

$$w = X + iY$$

$$\int_a^b \frac{|w'|}{Y}\,dt$$

$$w' = i\left[\frac{-z'(1+z) - (1-z)z'}{(1+z)^2}\right]$$

$$= \frac{-2iz'}{(1+z)^2}$$

$$|w'| = \frac{2|z'|}{|1+z|^2}$$

$w = x + iy$

$$\frac{w - \bar{w}}{2i}$$

$$w = i\,\frac{1-z}{1+z} \qquad \bar{w} = (-i)\,\frac{1-\bar{z}}{1+\bar{z}}$$

$$w - \bar{w} = i\left[\frac{1-z}{1+z} + \frac{1-\bar{z}}{1+\bar{z}}\right]$$

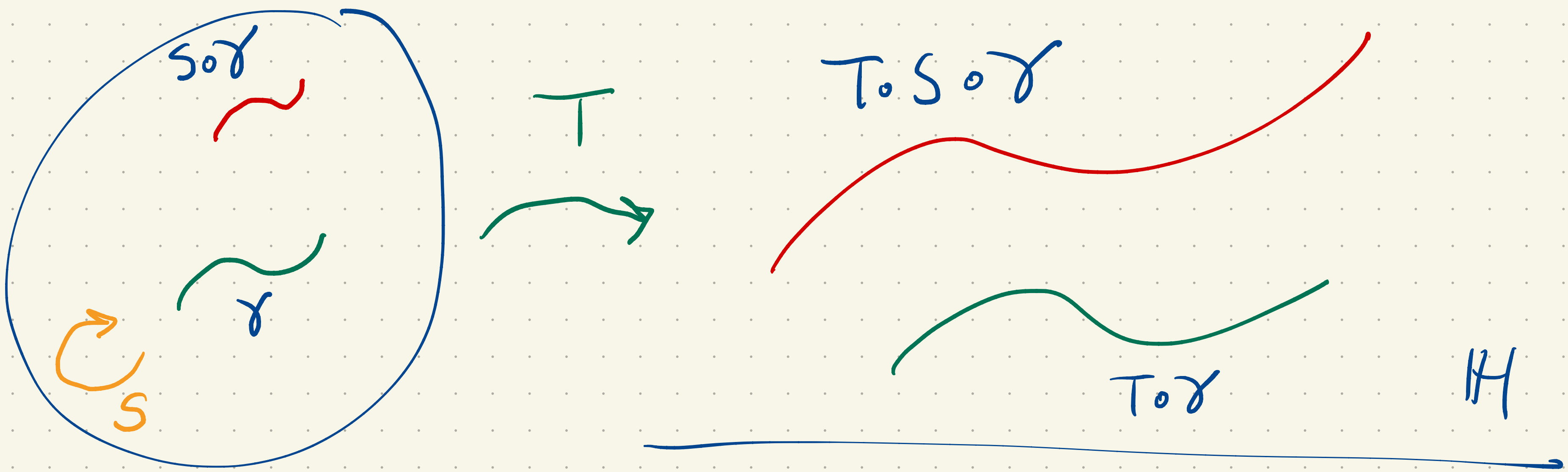
$$= i\left[\frac{(1-z)(1+\bar{z}) + (1-\bar{z})(1+z)}{|1+z|^2}\right]$$

$$= 2i\left[\frac{1-|z|^2}{|1+z|^2}\right]$$

$$\frac{w-\bar{w}}{2i} = \frac{1-|z|^2}{|1+z|^2}$$

$$\frac{|w'|}{Y} = \frac{2|z'|}{1-|z|^2} \qquad L_H(z) = \int_a^b \frac{2|z'|}{1-|z|^2}\,dt$$

$$\int_a^b \frac{|w'(t)|}{Y(t)}\,dt$$



$$L_H(\gamma) = L_H(T\circ\gamma)$$

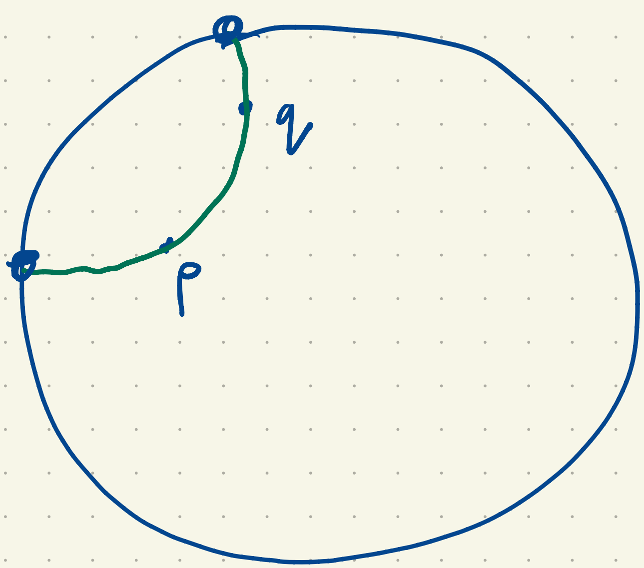
$$L_H(S\circ\gamma) = L_H(T\circ S\circ\gamma)$$

$$T\circ S\circ\gamma = \underbrace{(T\circ S\circ T^{-1})}_{\text{half plane hyp. transf.}}\circ(T\circ\gamma)$$

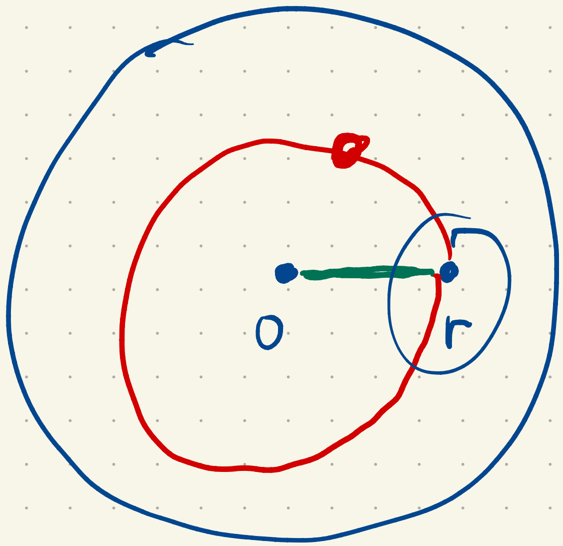
$$L_H(T \circ S \circ \gamma) = L_H(T \circ \gamma)$$

$$L_H(\gamma) = L_H(S \circ \gamma)$$

ball model
hyp transf.



$d_H(p, q)$



$z(t) = t \qquad 0 \le t \le r$

$$\int_0^r \frac{2|z'|}{1-|z|^2}\,dt = \int_0^r \frac{2}{1-t^2}\,dt$$

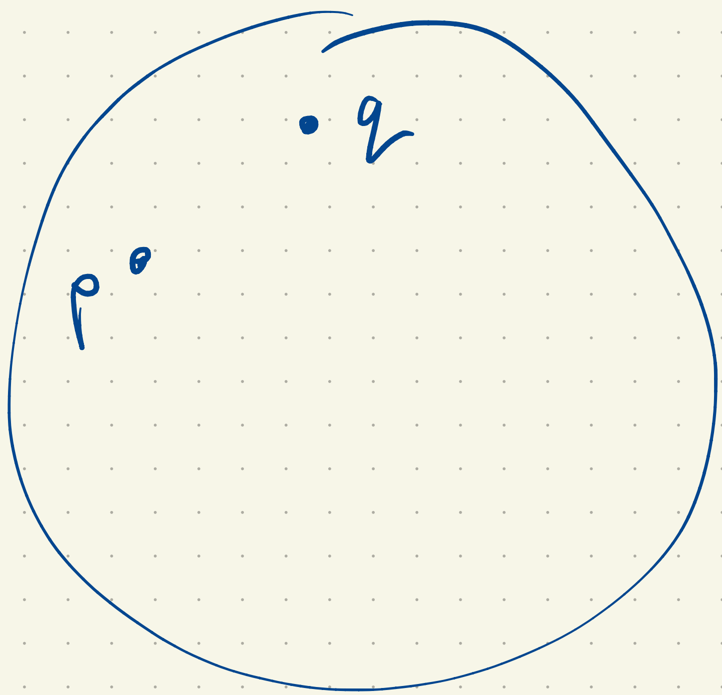
$$= \int_0^r \frac{1}{1+t} + \frac{1}{1-t}\,dt$$

$$= \ln(1+t)\Big|_0^r - \ln(1-t)\Big|_0^r$$

$$= \ln\left(\frac{1+r}{1-r}\right)$$

$$d_H(0,r) = \ln\left(\frac{1+r}{1-r}\right)$$

$$d_H(0, z) = \ln\left(\frac{1+|z|}{1-|z|}\right)$$



$$d_H(p, q)$$

$$p \to 0$$

$$S(z) = \frac{z-p}{1-z\bar{p}}$$

$$S(p) = 0$$

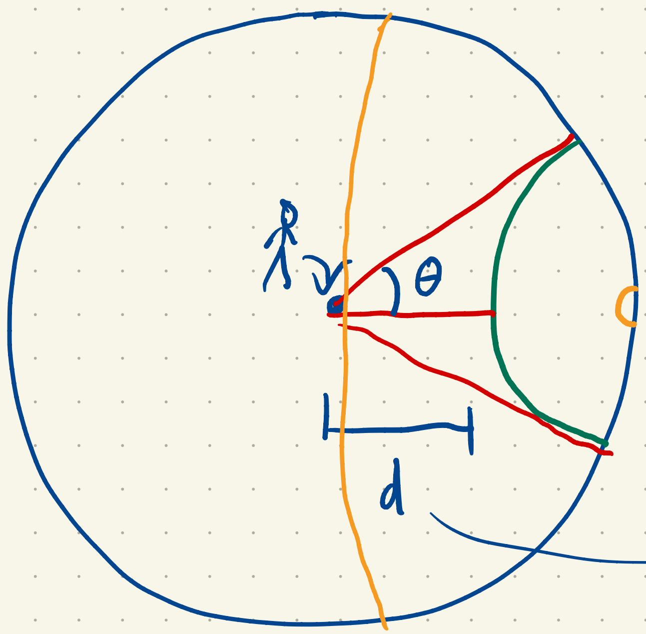
$$S(q) = \frac{q-p}{1-q\bar{p}}$$

$$d_H(p,q) = d_H(S(p), Sq))$$

$$= d_H(0, s_q)$$

$$= \ln\left(\frac{1 + |s_q|}{1 - |s_q|} \right)$$

$$= \ln\left(\frac{1 + \left|\frac{q-p}{1-q\bar{p}}\right|}{1 - \left|\frac{q-p}{1-q\bar{p}}\right|} \right)$$



$$\ln\left(\frac{1+r}{1-r}\right)$$

dist from me to line

Lobachevski's Formula.

$$e^{-d} = \tan\left(\frac{\theta}{2}\right)$$

$d \to 0$ $\theta \to \frac{\pi}{2}$