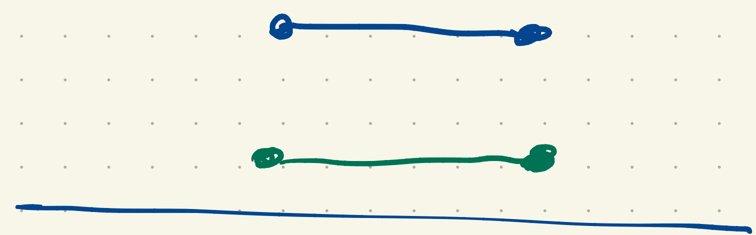
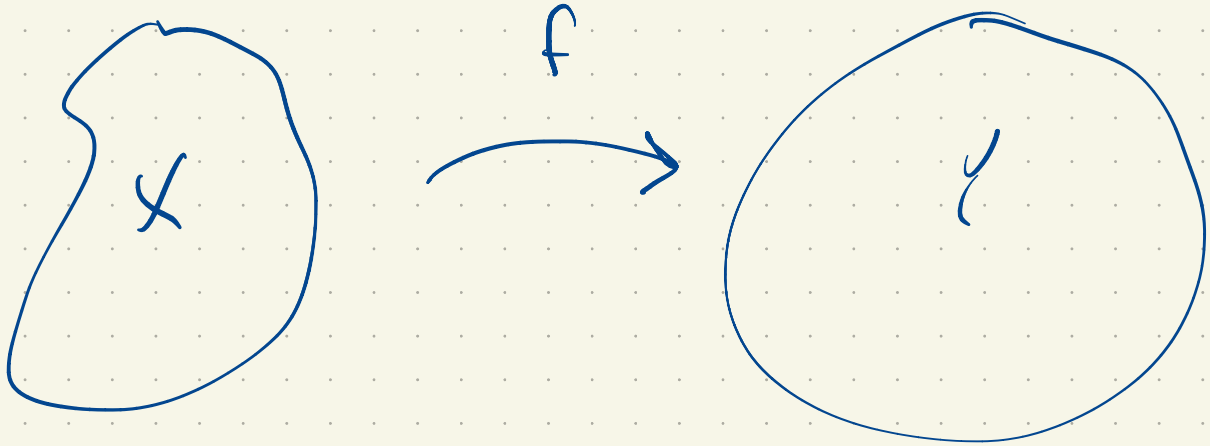
$$\sum_{k=0}^{n-1} |\gamma'(t_k)|\,\Delta t \xrightarrow{n\to\infty} \int_a^b |\gamma'(t)|\,dt$$

$$z(t) = x(t) + i\,y(t)$$

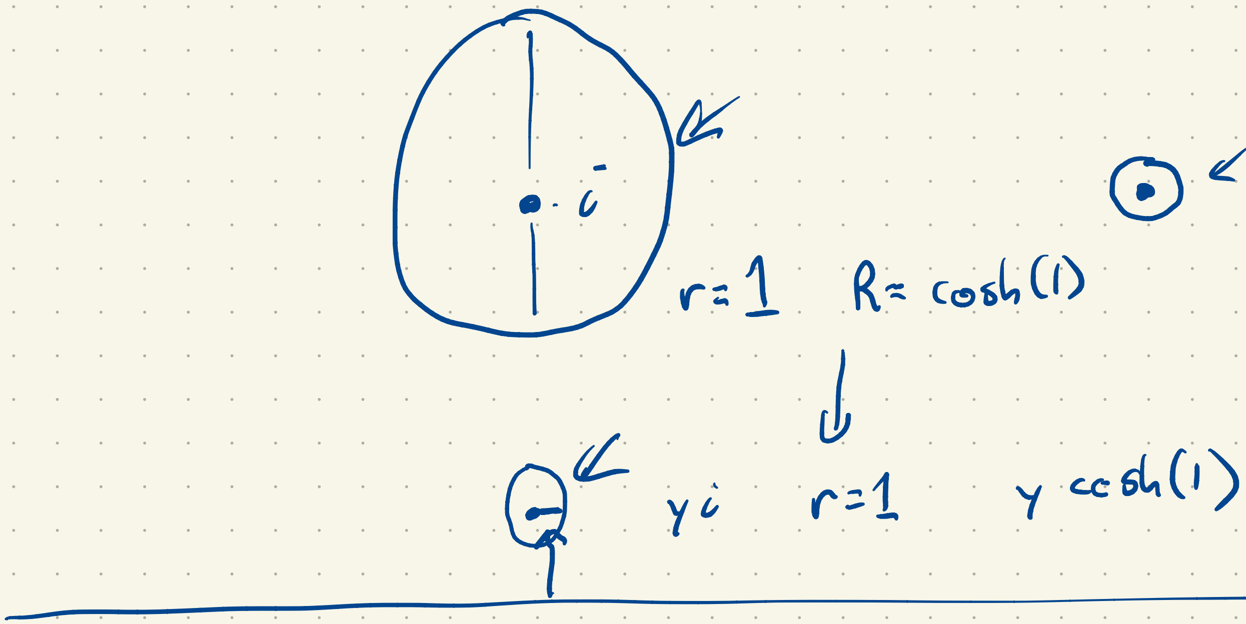
$\mathbb{H}$

$$\int_a^b \frac{|z'(t)|}{y(t)}\,dt$$





f is onto iff $\forall\, y \in Y \;\exists\, x \in X$ with $f(x) = y$.



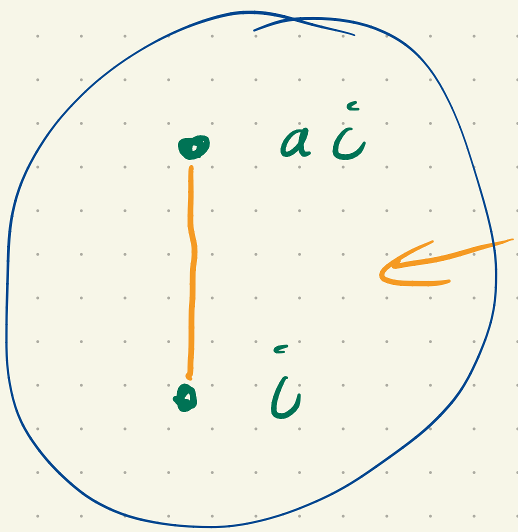
hyperbolic radius ε is roughly the circle centred at i with euclidean radius ε.
(ε small)

A tiny ball centred at yi with Euclidean radius ε

is roughly the hyperbolic ball centred at yi with

hyperbolic radius $\underline{\varepsilon/y}$ $\qquad$ $\underline{\varepsilon y}$

$$\frac{d}{dt} a^t$$



arc length

$$z(t) = e^{\ln(a)t}\, i$$

$$z(0) = i \qquad 0 \le t \le 1$$

$$z(1) = ai$$

$$z'(t) = \ln(a)\, e^{\ln(a)t}\, i$$

$$y(t) = e^{\ln(a)t}$$

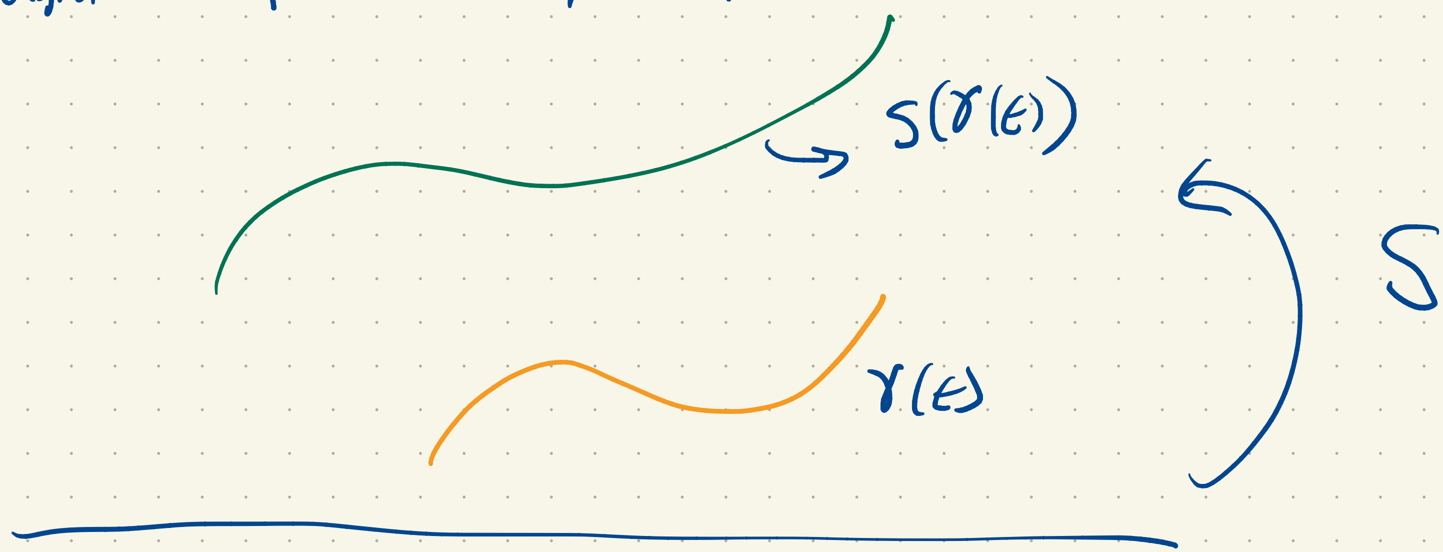
$$d_H(i, ai) = \left|\ln\left(\frac{ai}{i}\right)\right| = |\ln(a)|$$

$$|z'(t)| = |\ln(a)| e^{\ln(a) t}$$

$$\frac{|z'(t)|}{y(t)} = |\ln(a)|$$

$$\int_0^1 \frac{|z'(t)|}{y(t)} \, dt = \int_0^1 |\ln(a)| \, dt = |\ln(a)|$$

Arclength is preserved by hyperbolic transformations.



$l_H(\gamma)$ $\quad \gamma : [a,b] \to \mathbb{H}$

$\hookrightarrow \displaystyle\int_a^b \frac{|\gamma'(t)|}{y(t)}\,dt$

$l_H(\gamma) = l_H(S \circ \gamma)$

$z \equiv \gamma$

$w(t) = S(z(t))$

$w'(t) = ?$

$S(z) = \dfrac{az+b}{cz+d}$

a, b, c, d are real

$\boxed{ad - bc = 1}$

$$w(t) = \frac{a\,z(t) + b}{c\,z(t) + d}$$

$$\frac{d}{dt} w(t) = \frac{a z'(cz+d) - (az+b) c z'}{(cz+d)^2}$$

$$= \frac{(ad-bc) z'}{(cz+d)^2} = \frac{z'}{(cz+d)^2}$$

$$w(t) = X(t) + i Y(t) \qquad Y(t) = \frac{w(t) - \bar{w}(t)}{2i}$$

$$\frac{|w'(t)|}{Y(t)}$$

$$Y(t) = \frac{1}{2i}\left[\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right]$$

$$= \frac{1}{2i} \frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{1}{2i} \frac{adz + bc\bar{z} - ad\bar{z} - bcz}{|cz+d|^2}$$

$$= \frac{1}{2i} \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2}$$

$$= \frac{1}{|cz+d|^2} \frac{z-\bar{z}}{2i} \qquad z = x(t) + i y(t)$$

$$= \frac{1}{|cz+d|^2} y(t)$$

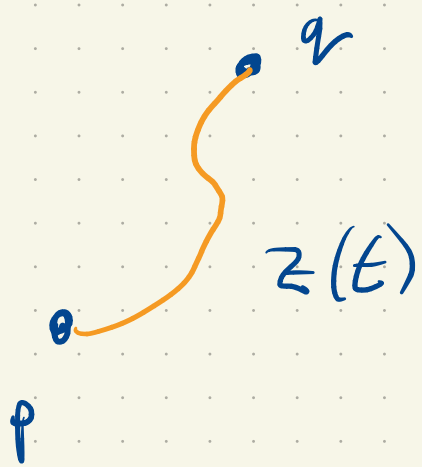
$$Y(t) = \frac{1}{|cz+d|^2}\, y(t)$$

$$w'(t) = \frac{z'}{(cz+d)^2} \qquad |w'| = \frac{|z'|}{|cz+d|^2}$$

$$\frac{|w'|}{Y} = \frac{|z'|}{|cz+d|^2}\,\frac{|cz+d|^2}{y} = \frac{|z'|}{y}$$

$$\int_a^b \frac{|w'(t)|}{Y(t)}\,dt = \int_a^b \frac{|z'(t)|}{y(t)}\,dt$$

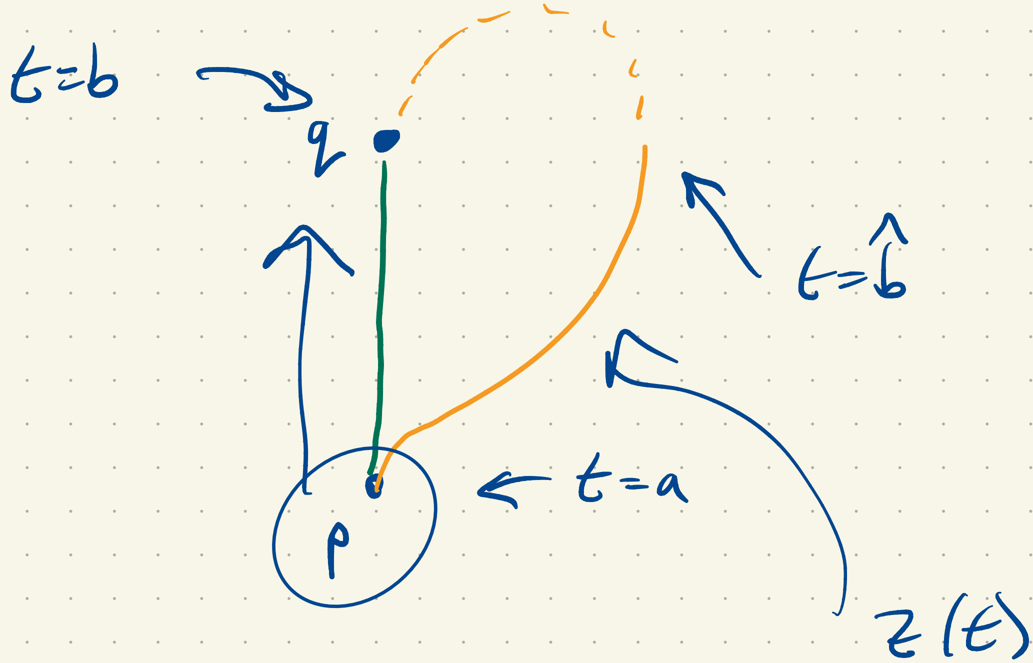
arclength is preserved.



$$z(a) = p$$

$$z(b) = q$$

$$d_H(p,q) \leq \underbrace{\int_a^b \frac{|z'(t)|}{y(t)}\,dt}_{\text{arclength}}$$



$$|z'| = \sqrt{(x')^2+(y')^2} \geq |y'|$$

$$\int_a^b \frac{|z'(t)|}{y(t)}\,dt \geq \int_a^{\hat{b}} \frac{|z'(t)|}{y(t)}\,dt$$

$$\geq \int_a^{\hat{b}} \frac{|y'(t)|}{y(t)}\,dt$$

$$\geq \int_a^{\hat{b}} \frac{y'(t)}{y(t)}\, dt \qquad u = y(t)$$

$$du = y'(t)\, dt$$

$$= \ln\left(y(\hat{b}) / y(a) \right)$$

$$= \left| \ln\left(y(\hat{b}) / y(\dot{a}) \right) \right|$$

$$= \left| \ln\left(q / p \right) \right|$$

$$= d_H(p, q)$$

$$d_H(p,q) \le d_H(p,r) + d_H(r,q) \quad p,q,r \in H$$