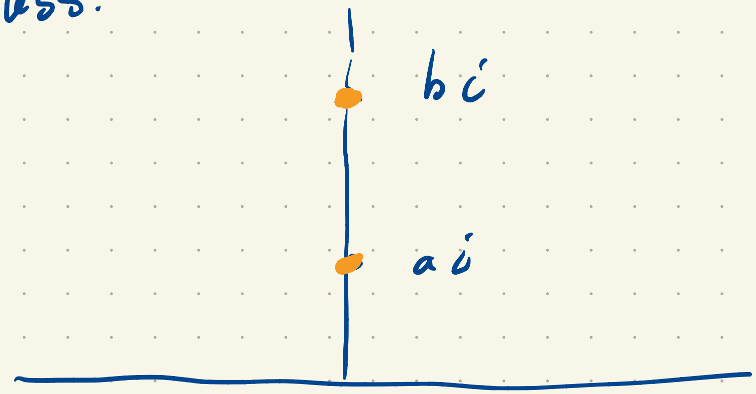
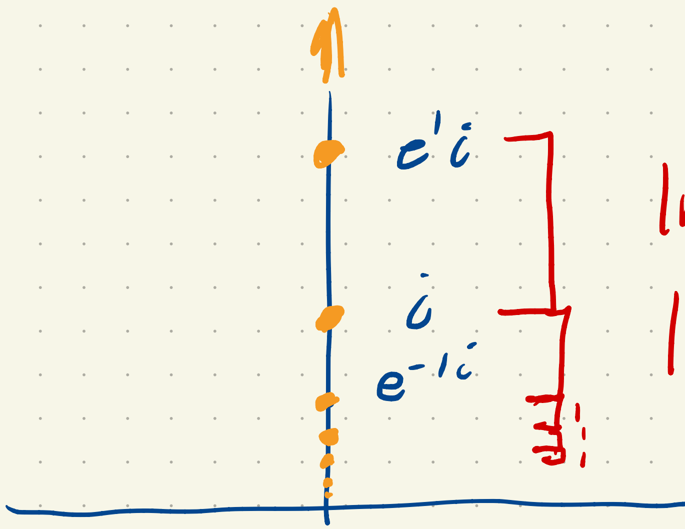
Last class:



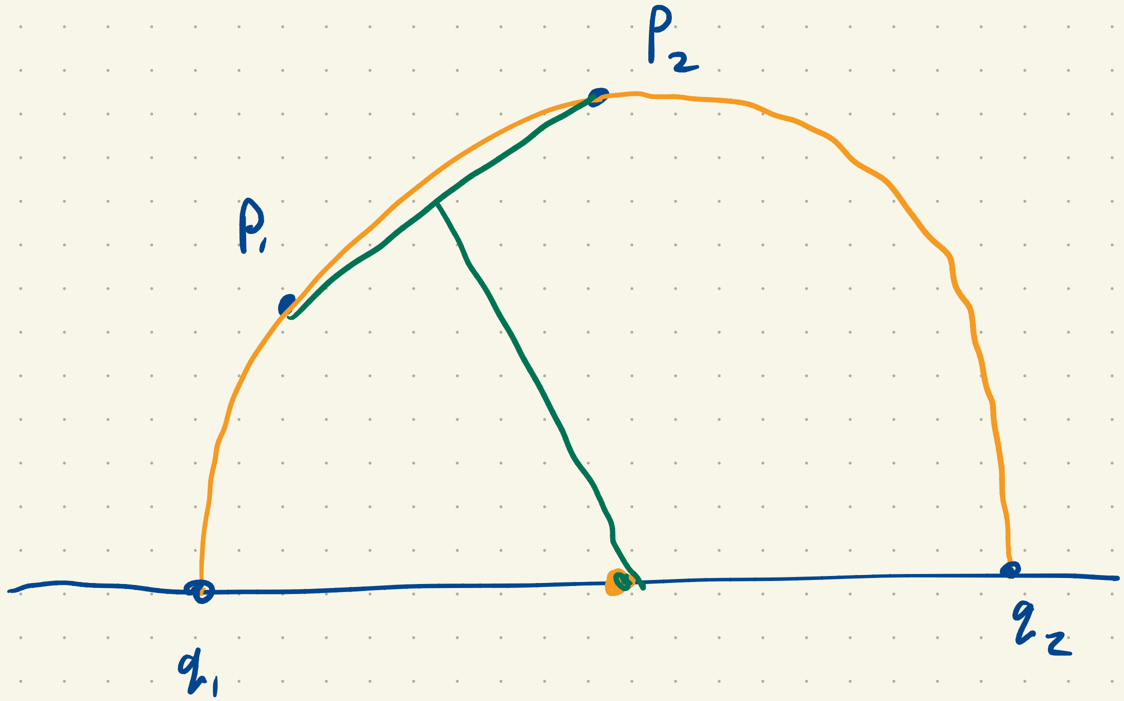
$$d_H(ai, bi) = \left| \ln\left(\frac{ai}{bi}\right) \right| = \left| \ln\left(\frac{a}{b}\right) \right|$$

$e^k i \qquad k \in \mathbb{Z}$



$\ln\left(e^1/e^0\right) = 1$

$\left|\ln\left(e^0/e^{-1}\right)\right| = 1$

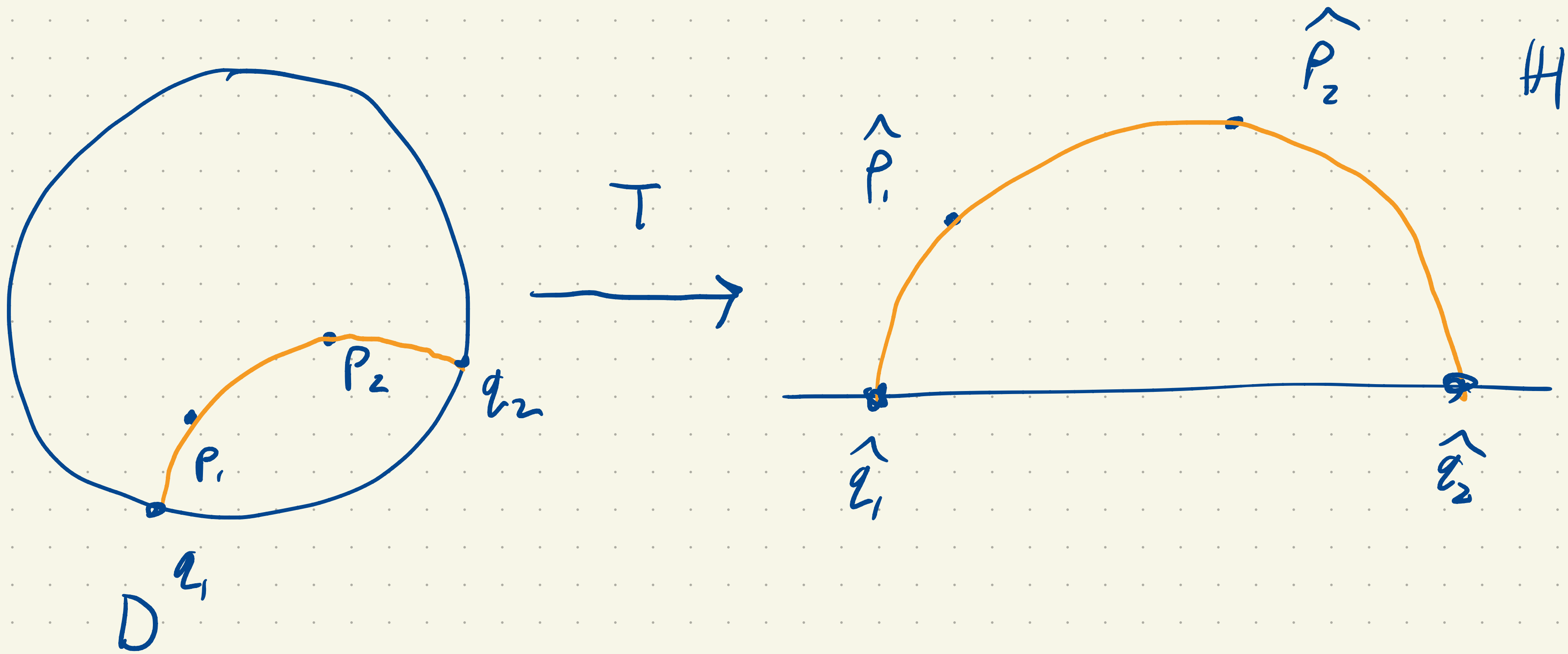


$$d_H(p_1, p_2) = \left| \ln\left((p_1, p_2, q_1, q_2) \right) \right|$$

upper half space

If $S : \mathbb{H} \to \mathbb{H}$ is a hyperbolic transformation then

$$d_H(p_1, p_2) = d_H(S(p_1), S(p_2))$$

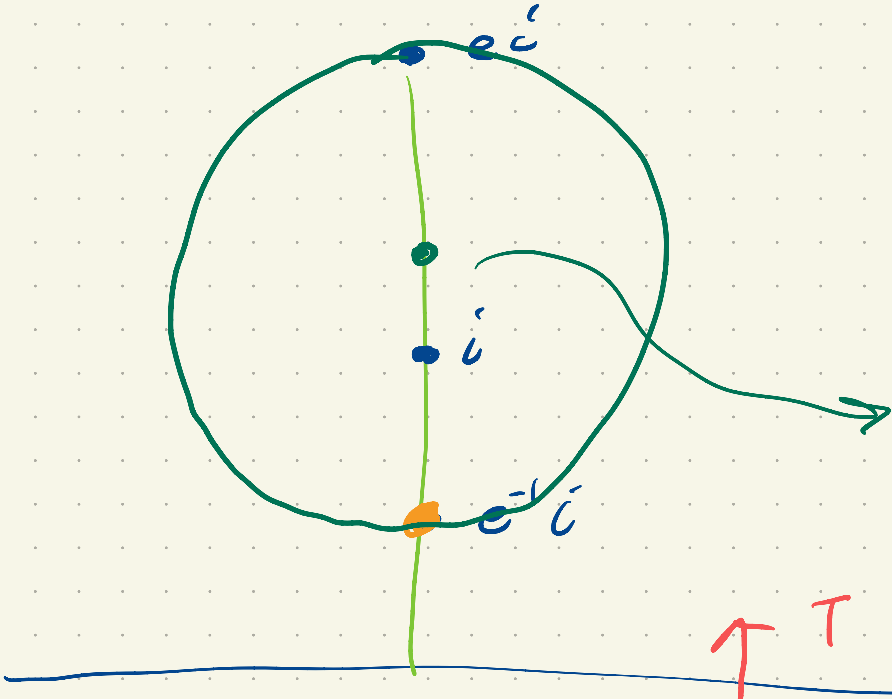


$$d_H(p_1, p_2) = \left| \ln\left((\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2) \right) \right|$$

$$= \left| \ln\left((Tp_1, Tp_2, Tq_1, Tq_2) \right) \right|$$

$$= \left| \ln\left((p_1, p_2, q_1, q_2) \right) \right|$$

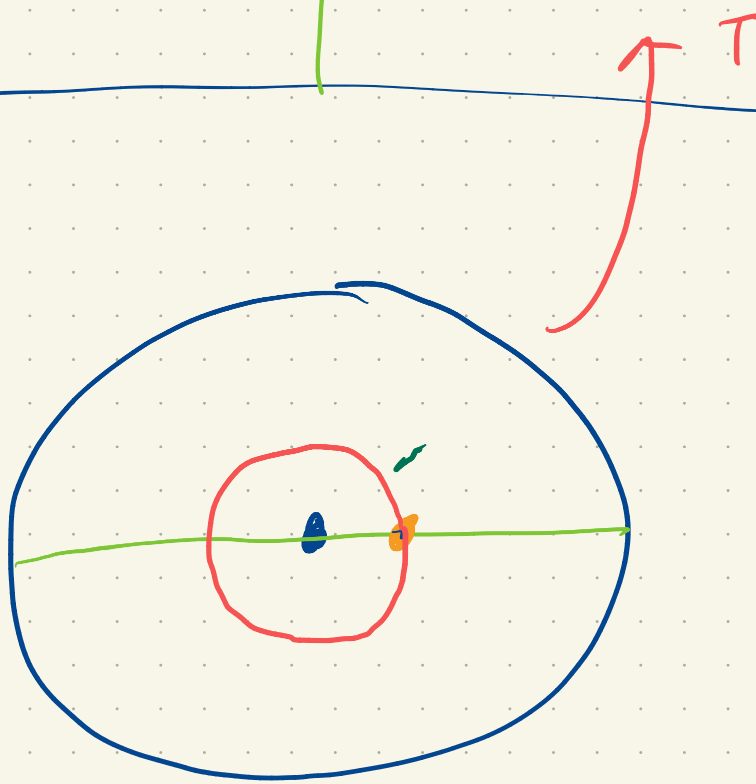
Circles: What are all the points of distance 1 from i in $\mathbb{H}$



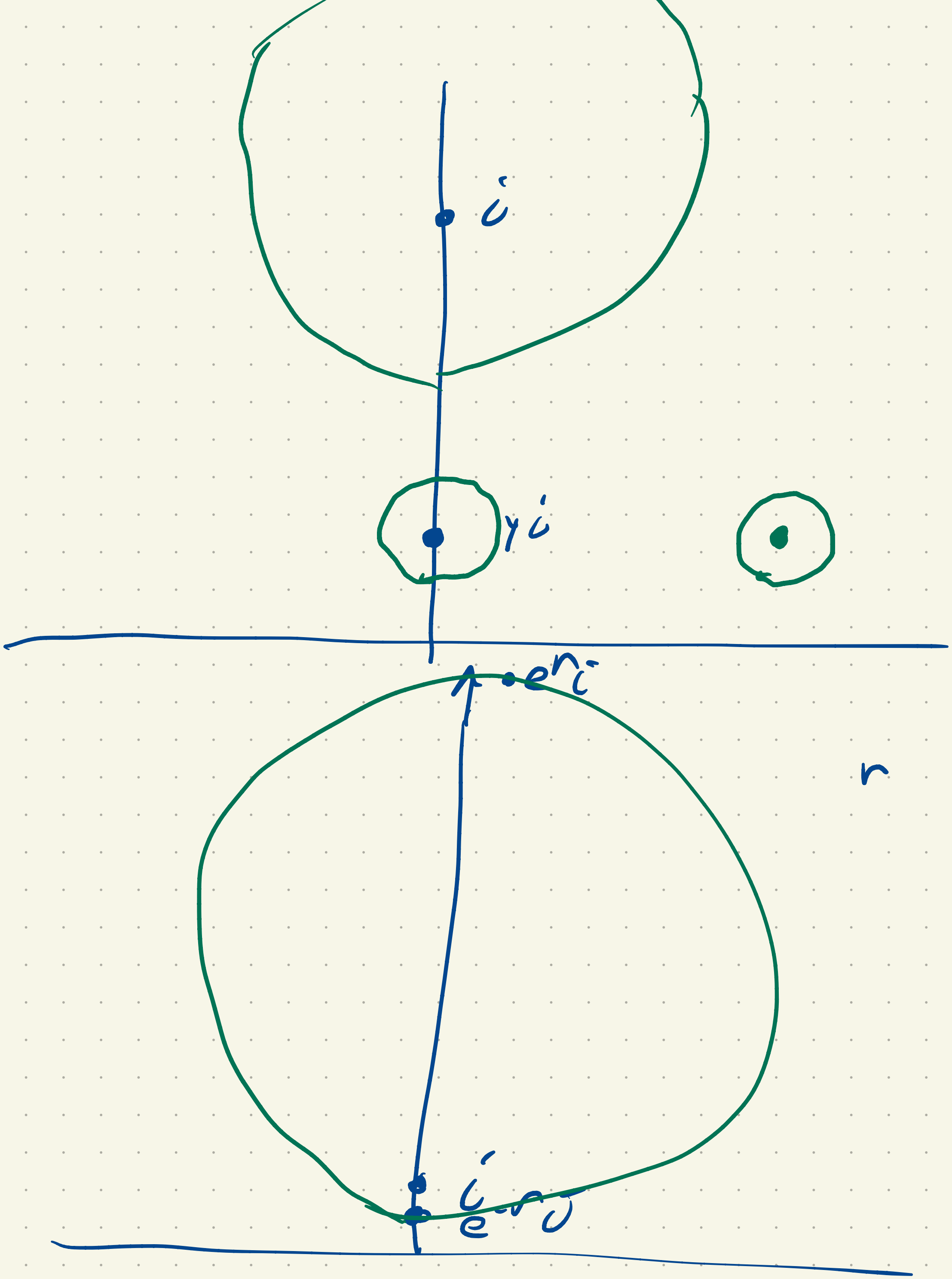
$$|z-i|=1$$

$$\cosh(x)=\frac{e^{x}+e^{-x}}{2}$$

$$\underbrace{\frac{e+e^{-1}}{2}}\, i \;\to\; \cosh(1)\, i$$



$$\lambda\ \frac{z-p}{1-z\bar{p}}$$



$z \mapsto yz$

is a hyp. transf.

$z \mapsto z + a \quad a \in \mathbb{R}$

r

ai

$$\left| \ln(ai/i) \right| = r$$

$$\ln(a) = \pm r$$

$$a = e^{\pm r}$$

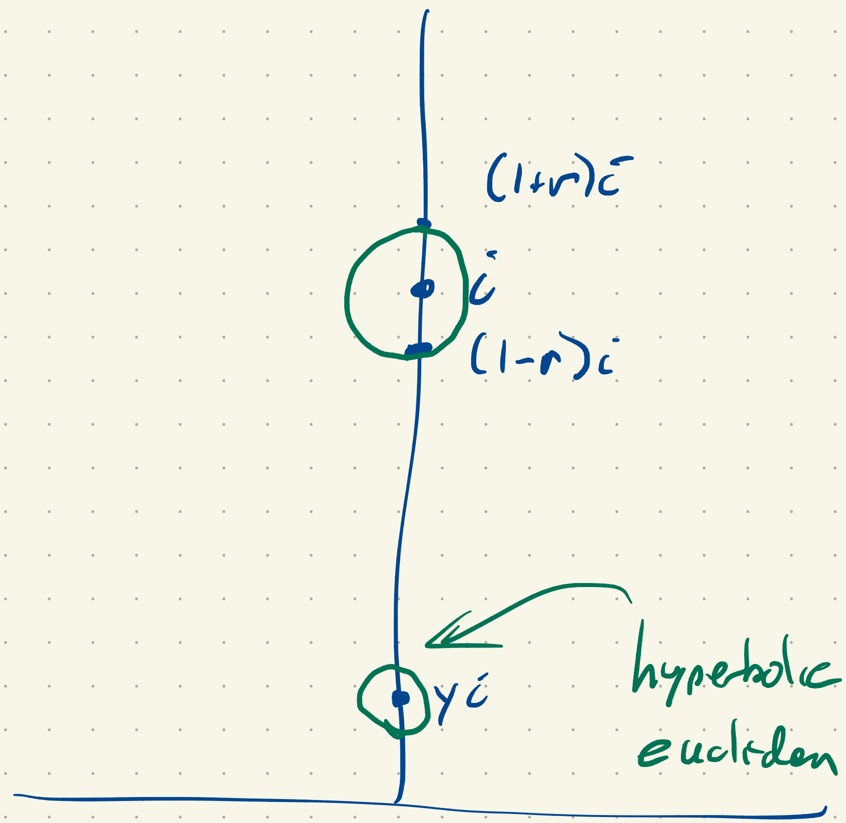
$$\frac{e^r + e^{-r}}{2} i$$

$\cosh(r)\, i$

If r is small $\quad e^{\pm r} \approx 1 \pm r$

$$= 1 \pm r + O(r^2)$$

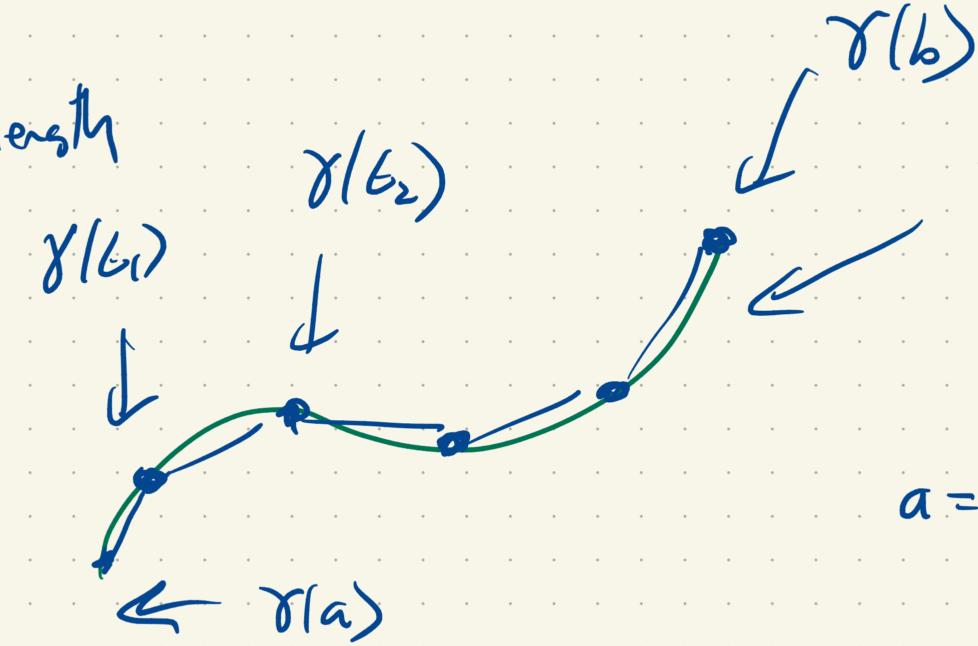
$$\cosh(r) = 1 + O(r^2)$$



The hyperbolic circle is roughly the Euclidean circle of radius r

$$R/y = r$$

Arclength



$\gamma(t) = (x(t), y(t))$

$a \le t \le b$

$a = t_0 < t_1 < \cdots < t_n = b$

$t_k = a + k\,\Delta t$

$\Delta t = \frac{b-a}{n}$

$\gamma(t_k + \Delta t)$

$\approx \gamma(t_k) + \gamma'(t_k)\,\Delta t$

$\gamma(t_k)$

$$\left|\gamma(t_k+\Delta t) - \gamma(t_k)\right| = \left|\gamma'(t_k)\right|\Delta t$$

$$\sum_{k=0}^{n-1} |\gamma'(t_k)|\,\Delta t \xrightarrow{n\to\infty} \int_a^b |\gamma'(t)|\,dt$$

$$z(t) = x(t) + i\,y(t)$$

$\mathbb{H}$

$$\int_a^b \frac{|z'(t)|}{y(t)}\,dt$$