$\uparrow$ iff

$a,b,c,d \in \mathbb{R}$

$$\frac{az+b}{cz+d} = \frac{az+b}{cz+d}\,\frac{c\bar{z}+d}{\underbrace{c\bar{z}+d}_{\overline{cz+d}}} = \frac{ac|z|^2 + bc\bar{z} + adz + bd}{|cz+d|^2}$$

$$= \underbrace{\frac{ac|z|^2+bd}{|cz+d|^2}}_{\uparrow} + \underbrace{\frac{adz+bc\bar{z}}{|cz+d|^2}}$$

$z = x+iy$

$$\frac{(adx+bcx)+iy(ad-bc)}{|cz+d|^2}$$

imaginary part is $\quad iy \dfrac{(ad-bc)}{|cz+d|^2}$

$\underbrace{\text{upper half plane}}_{y>0}$ sends to upper half plane $\iff$ $\underbrace{ad-bc>0}$

$z \to -\bar{z}$

$$Tz = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{R}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$ad-bc>0$

WLOG $ad-bc=1$

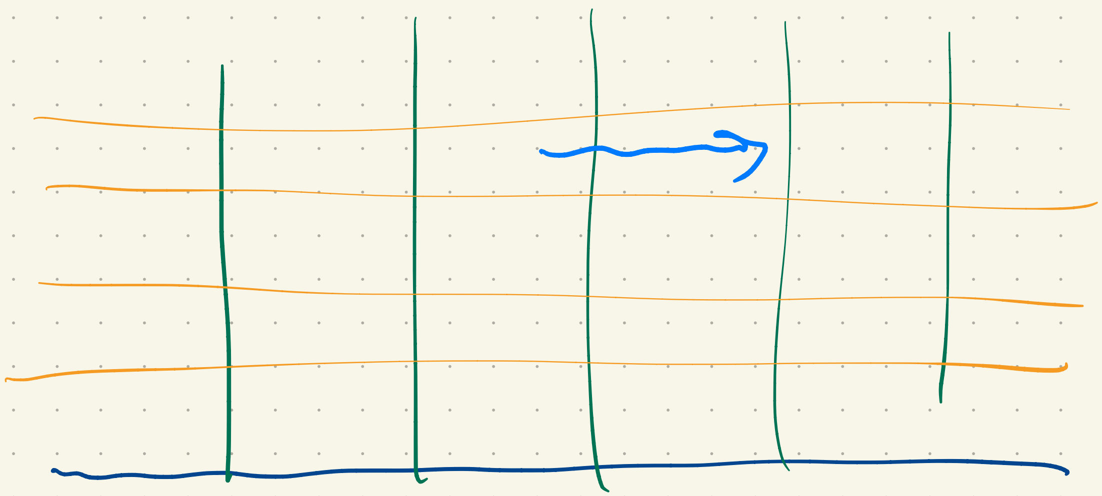
$$Tz = \lambda \frac{z-p}{1-\bar{p}z} \qquad \lambda \in S^1,\ p \in D$$

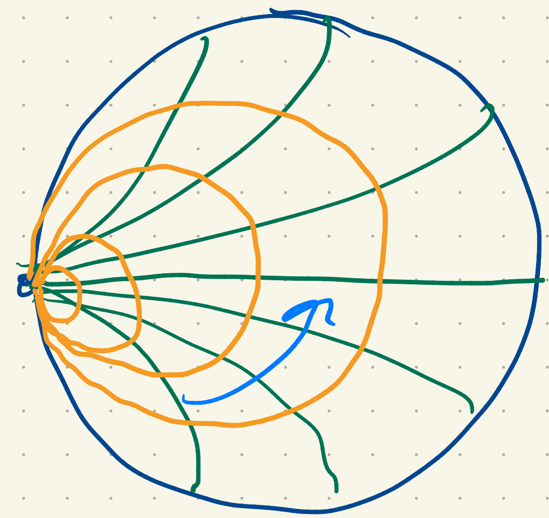
$$Tz = \frac{az+b}{cz+d}$$

$a = 1,\ b,\ c = 0,\ d = 1$

$ad - bc = 1 \cdot 1 - b \cdot 0 = 1$

$$Tz = z + b \quad \mathbb{R}$$

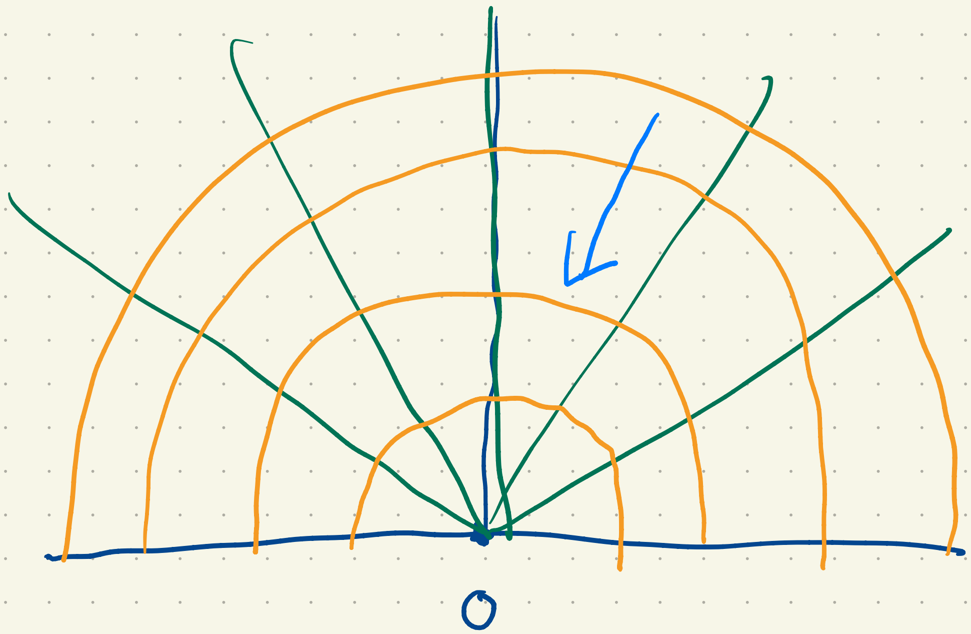


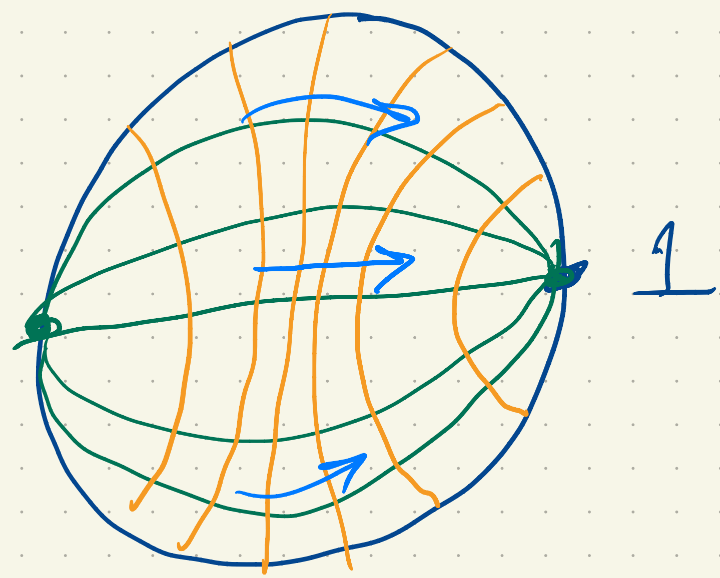


$Tz = az \quad \text{if} \quad a > 0$

$a,\ b = 0,\ c = 0,\ d = 1$

$ad - bc = ad = a > 0$





Translation

Rotations about i

$$Tz \mapsto \frac{az+b}{cz+d}$$

$$Ti = i$$

$$Ti = i$$

$$\frac{ai+b}{ci+d} = i$$

$$ai+b = -c+di$$

$a, b, c, d \in \mathbb{R}$

$$a = d$$

$$b = -c$$

$$Tz = \frac{(az+b)\lambda}{(-bz+a)\lambda}$$

$$\lambda^2(a^2+b^2) > 0 \qquad \lambda = \frac{1}{\sqrt{a^2+b^2}}$$

$$a^2+b^2 = 1$$

$$\begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

$$a = \cos(\theta/2) \qquad b = \sin(\theta/2)$$

$$\sin(\varphi+\pi) = -\sin(\varphi)$$

$$\cos(\varphi+\pi) = -\cos(\varphi)$$

$$\cos\left(\frac{\theta+2\pi}{2}\right) = \cos\left(\frac{\theta}{2}+\pi\right) = -\cos(\theta/2)$$

$$\sin\left(\frac{\theta+2\pi}{2}\right) = -\sin(\theta/2)$$

$$Tz = \frac{az+b}{-bz+a}$$

$\theta = \pi/2$

$\cos(\theta/2) = \cos(\pi/4) = 1/\sqrt{2}$

$\sin(\theta/2) = \sin(\pi/4) = 1/\sqrt{2}$

$$Tz = \frac{z+1}{-z+1}$$

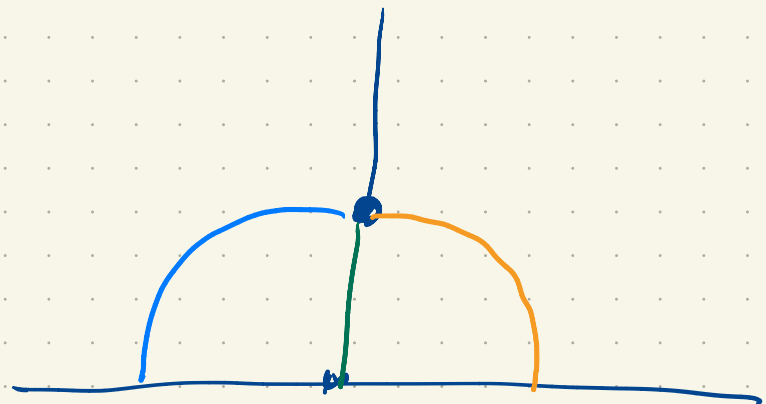
$T(i) = i$

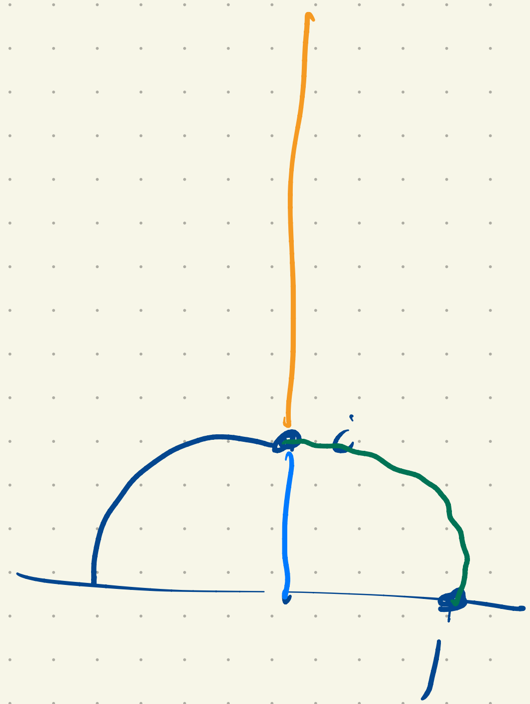
$T(0) = 1$

$T(1) = \infty$

$T(\infty) = -1$

$T(-1) = 0$





Exercise: do this again with $\theta = \pi$.