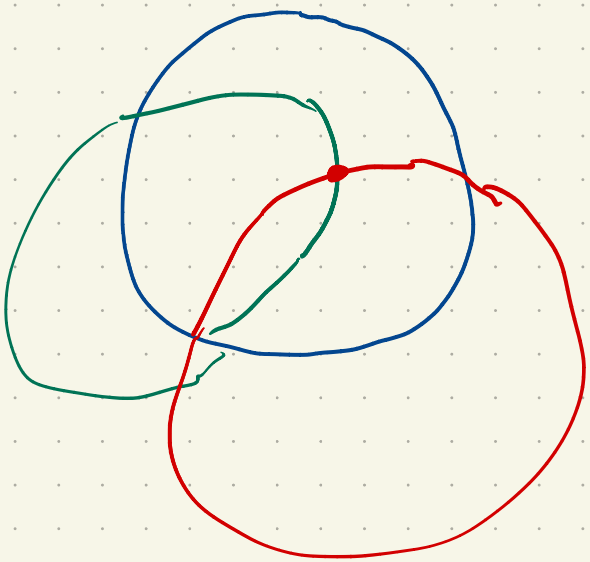
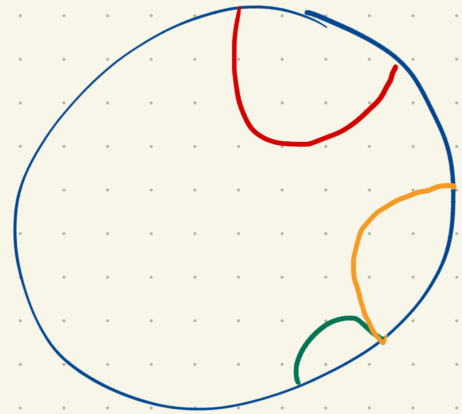
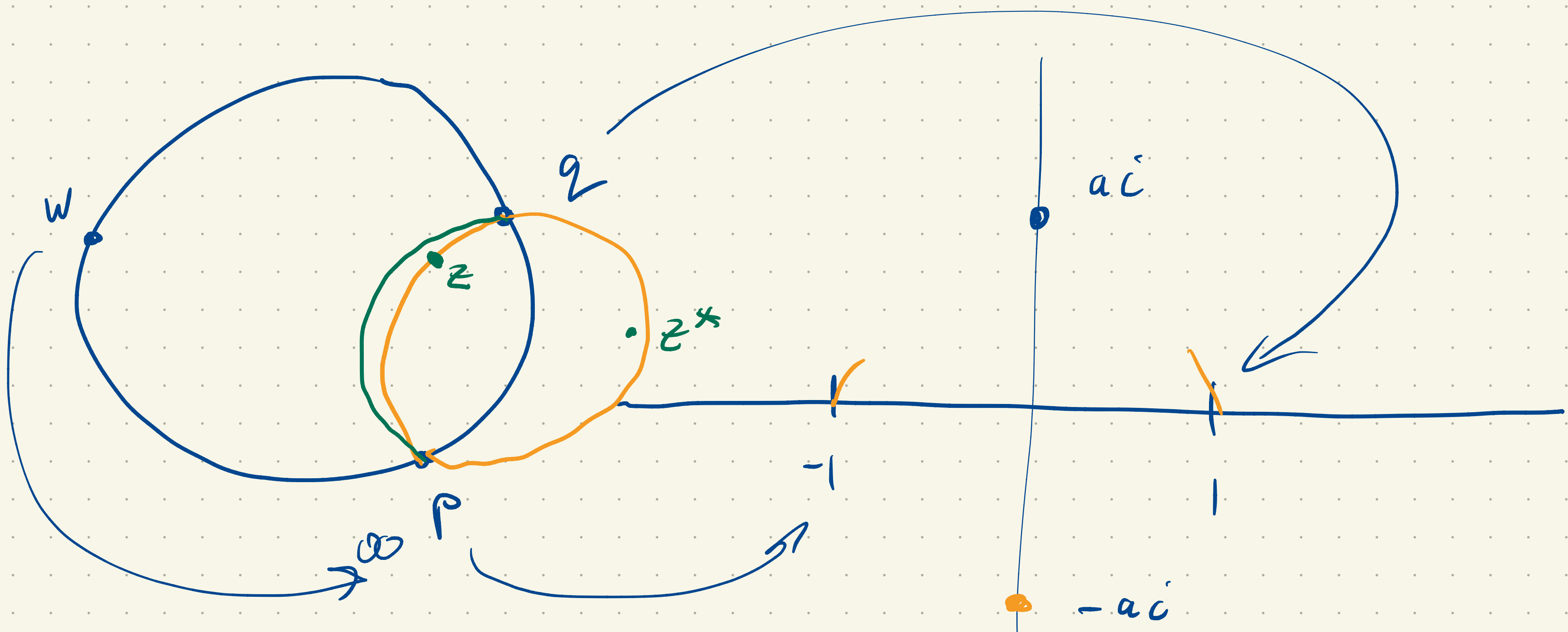




$a = 0, 1, -1$



$$(1, -1, ai, -ai) = \frac{1-ai}{1+ai} \, \frac{-1+ai}{-1-ai}$$

$$\vdots$$

$$= \frac{(1-ai)^4}{(1+a^2)^2}$$

$$(1-ai)^2 = (1-a^2) - 2ai$$

$$(1-ai)^4 = \left[(1-a^2)^2 - 4a^2\right] - 4a(1-a^2)\,i$$

$$a(1-a^2) = 0 \quad \Rightarrow \quad a = 0, \; a = \pm 1$$

Remark: one can show by similar techniques that given two ideal points there is a hyperbolic line passing through both.

(is at most one, by above, so it's unique!)

---

one point of intersection in D (and one outside S^1)

"not parallel"

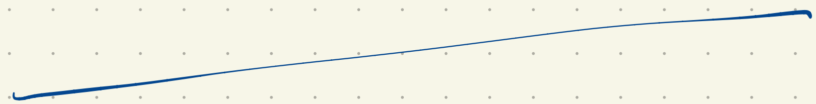
no intersections in D or on S^1 : hyperparallel

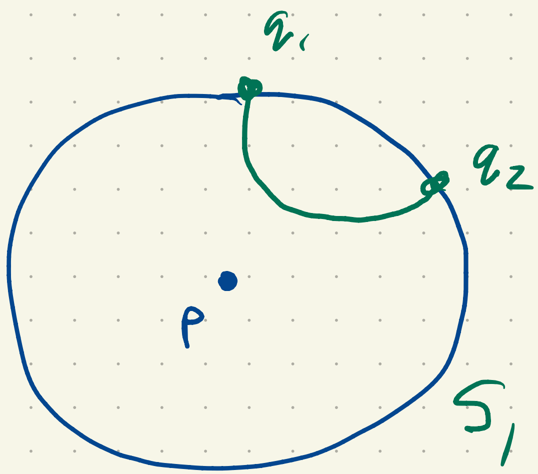
one intersection on S^1 (none in D) : parallel

Angle of parallelism.

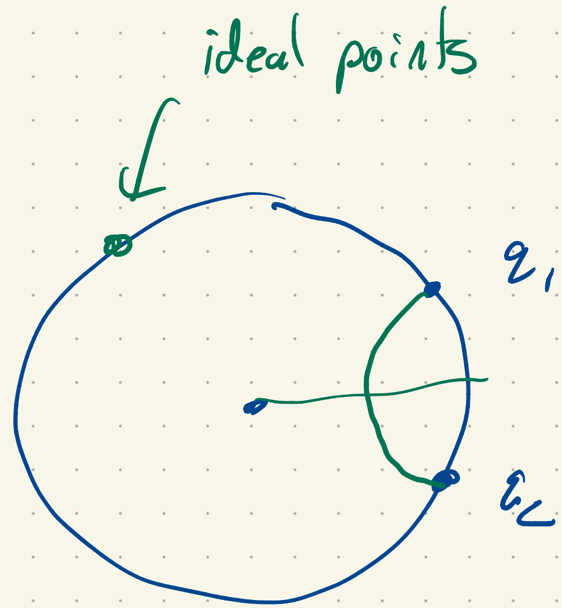
line: L

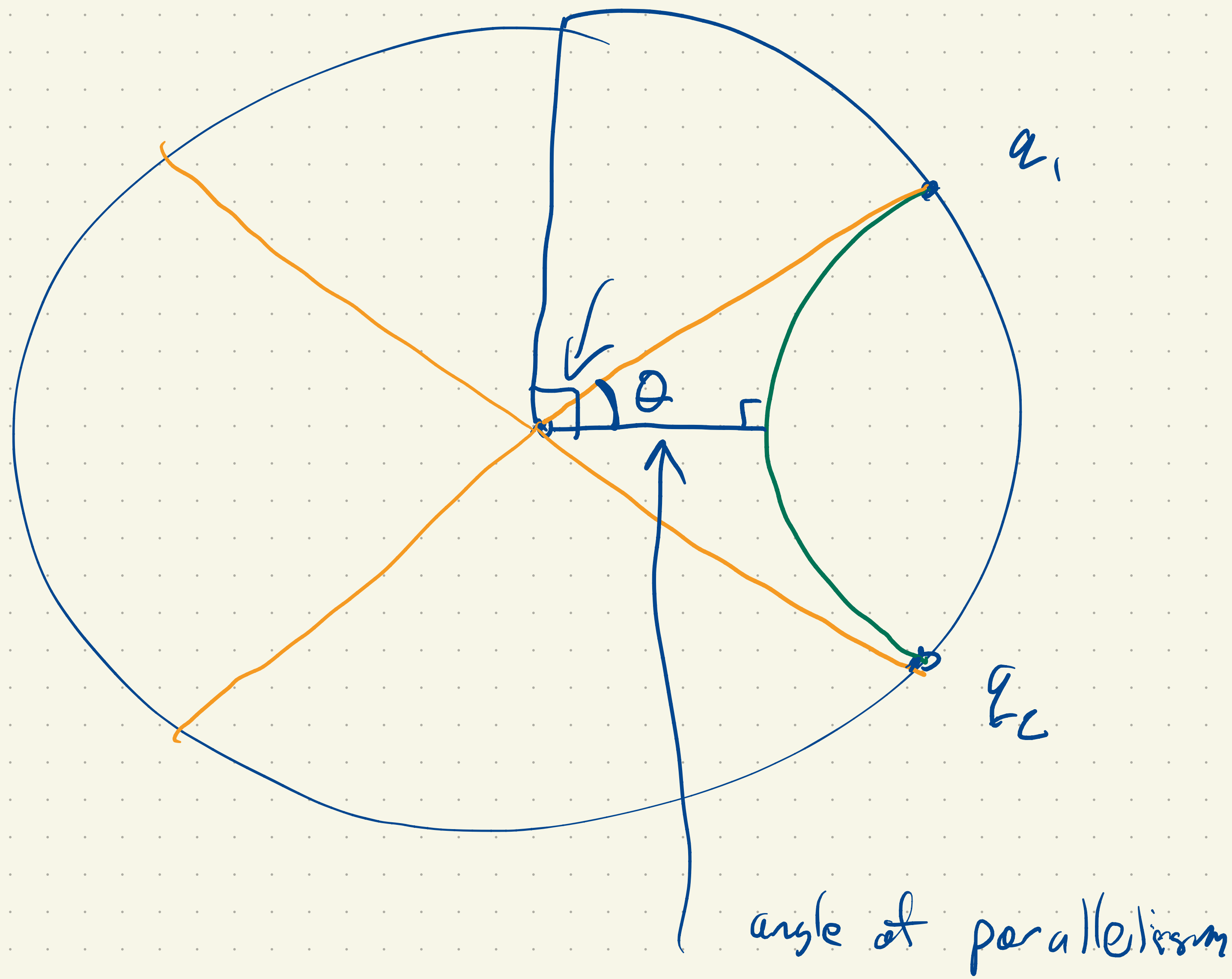
point p not on L.



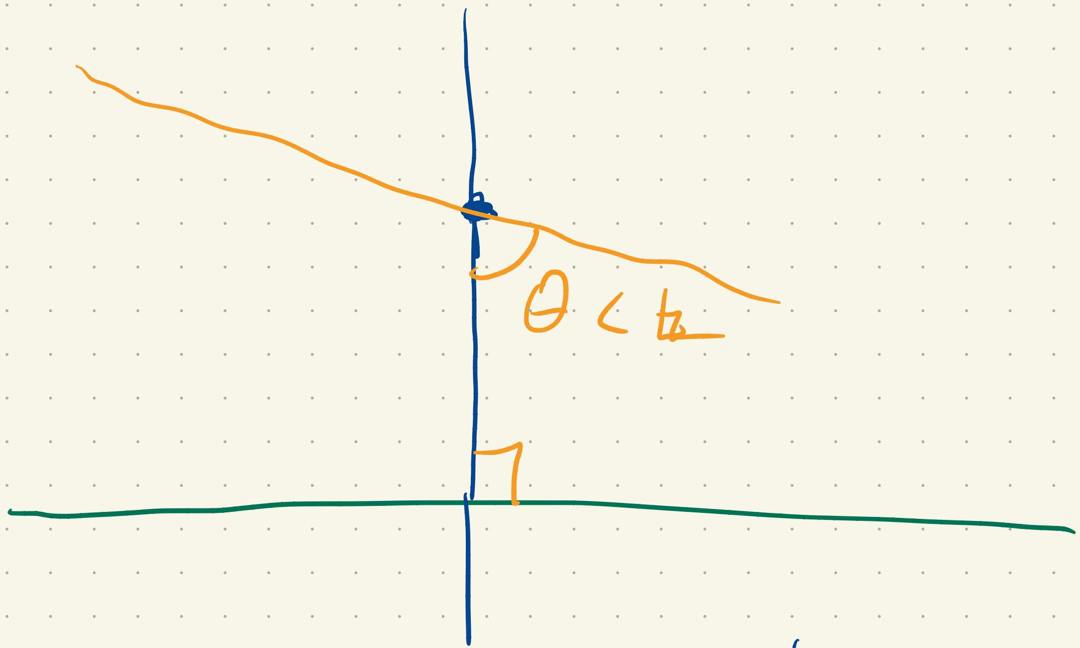


not antipodal

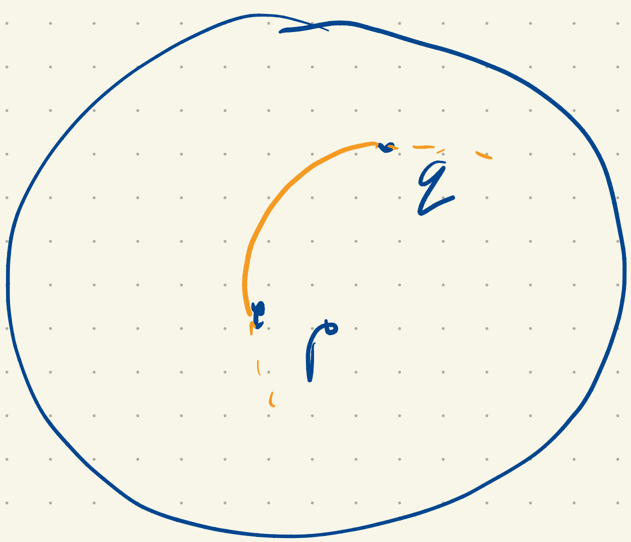




By construction θ is less than a right angle.



We have violated postulate 5.



Hyperbolic transformations.

Möbius transformations $\longrightarrow$ at most two (unless the id)

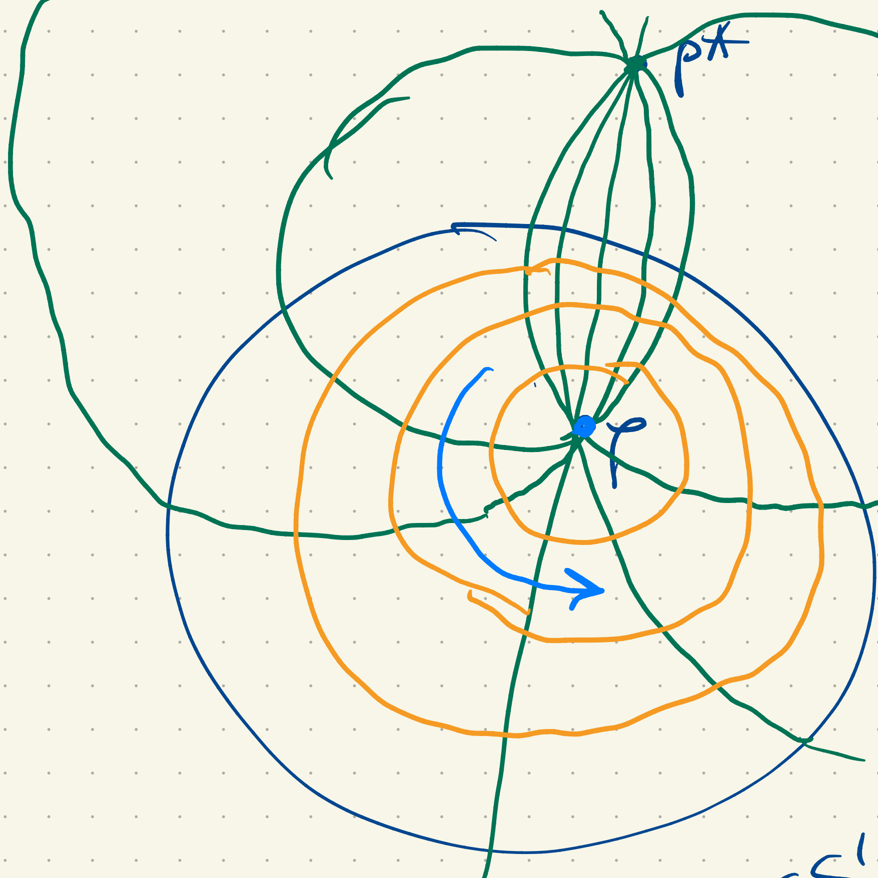
one is possible, but not none.

Suppose T is a hyperbolic transformation

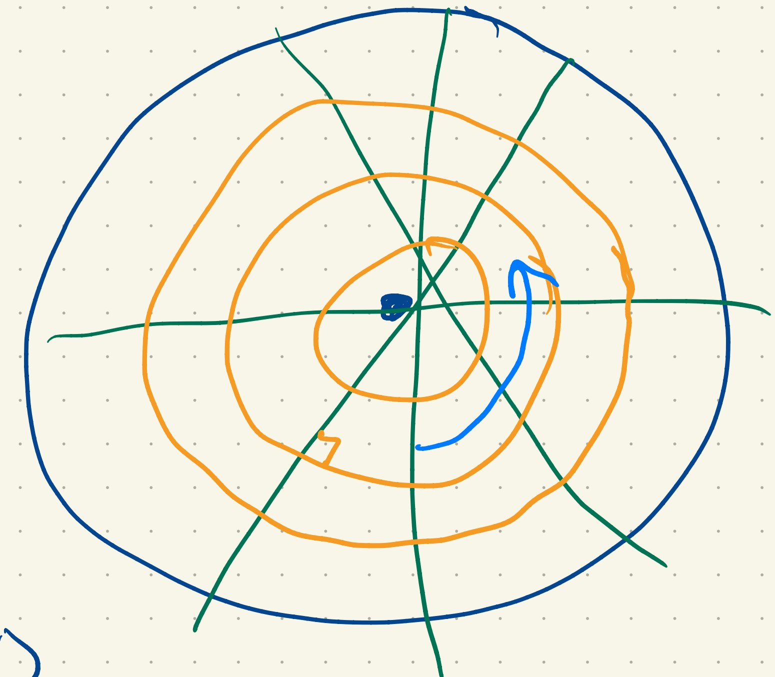
and $p \notin S^1$ is a fixed point.

$$T(p) = p$$

$$T(p^*) = \left(T(p)\right)^* = p^*$$



Hyperbolic Rotations



$$T(z) = \underset{\in S^1}{\lambda} \, \frac{z - \overset{\in D}{q}}{1 - \bar{q} z}$$

$$T(0) = 0$$

$$T(\infty) = \infty$$

$$T(0) = \frac{\lambda \cdot (-q)}{1} = 0$$

$$\Rightarrow$$

$$T(z) = \lambda z \qquad \lambda = e^{i\theta}$$