Is this a trans. group? Yes!

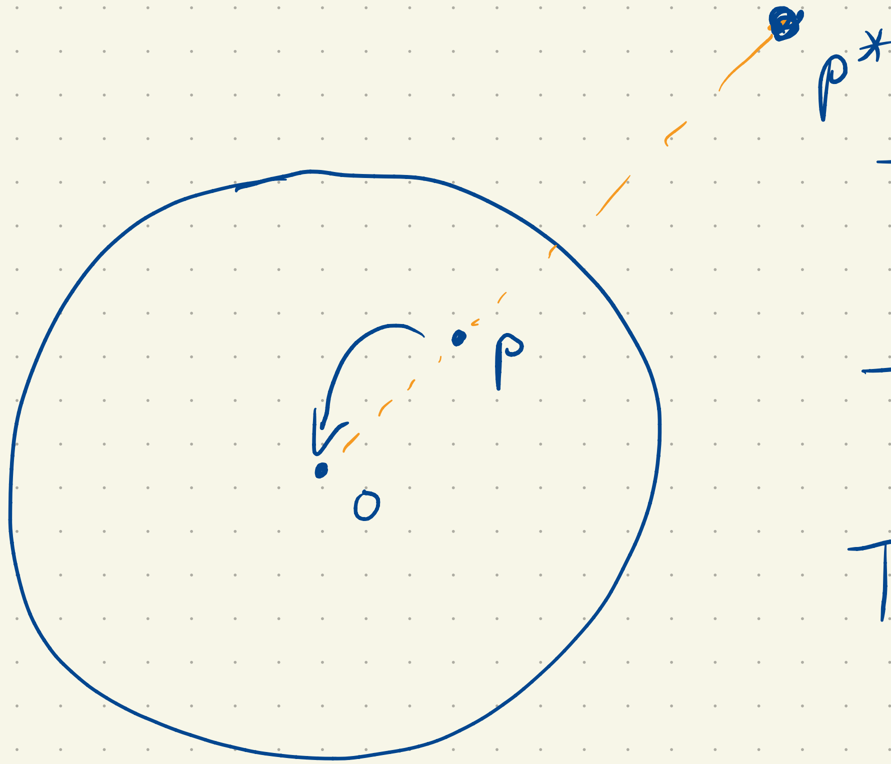
a) id ✓

b) closed under composition ✓

c) closed under inverses ✓

Who are these?

$T(D) = D$ implies there is some $p \in D$ with $Tp = 0$



$T: \mathbb{C}^+ \to \mathbb{C}^+$

$T|_D : D \to D$

$T(p) = 0$

$T(S^1) = S^1$

p^*

$$T(p^*) = (T(p))^* = (0)^* = \infty$$

$$p^* = \frac{p}{|p|^2} = \frac{p}{|p|} \cdot \frac{1}{|p|}$$

$\bar{p}^{-1}$

$$Tz = a\,\frac{z-p}{z-p^*} \qquad a \in \mathbb{C},\ a \neq 0$$

$$= a\,\frac{z-p}{z-\bar{p}^{-1}}$$

$$= -\bar{p}a\,\frac{(z-p)}{1-\bar{p}z}$$

$$= \lambda\,\frac{z-p}{1-\bar{p}z}$$

T $\quad z_1 \to 1$

$z_2 \to 0$

$z_3 \to \infty$

$$Tz = (z, z_1, z_2, z_3)$$

$$|T(1)| = 1 \qquad |T1| = \left|\lambda\,\frac{1-p}{1-\bar{p}\cdot 1}\right|$$

$$= |\lambda| \left| \frac{1-\rho}{1-\bar{\rho}} \right|$$

$$|a| = |\bar{a}|$$

$$= |\lambda| \left| \frac{1-\rho}{\overline{1-\rho}} \right|$$

$$= |\lambda| \frac{|1-\rho|}{|\overline{1-\rho}|}$$

$$= |\lambda|$$

$$|\lambda| = 1$$

$$T(z) = e^{i\theta} \frac{z-\rho}{1-\bar{\rho}z}$$

(*)

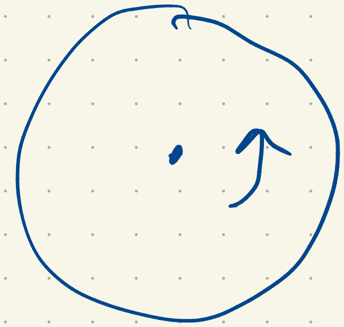
<u>If</u> T is a hyperbolic transformation then T has this form

$\rho \in D$

$\theta \in \mathbb{R}$

$T(z) = z$ iff $p = 0$ $\theta = 0$

If $p = 0$, this is a "Euclidean" rotation



For the converse (every T of the form $(*)$ is a hyperbolic transformation.

Exercise: a) Given such a T, show

$$T^{-1}(w) = \lambda^{-1} \frac{z - q}{1 - z\bar{q}} \qquad q = -\lambda p \in D$$

b) If $0 \le c, d < 1$ then $1 + cd > c^2 + d^2$

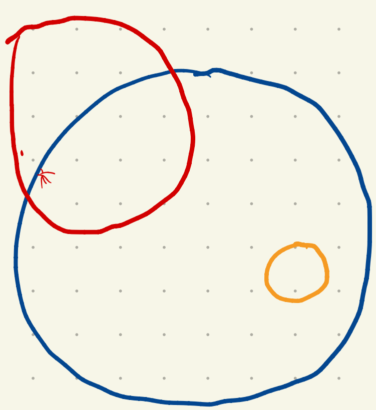
c) Show if $|z| < 1$ then $|Tz|^2 < 1$.

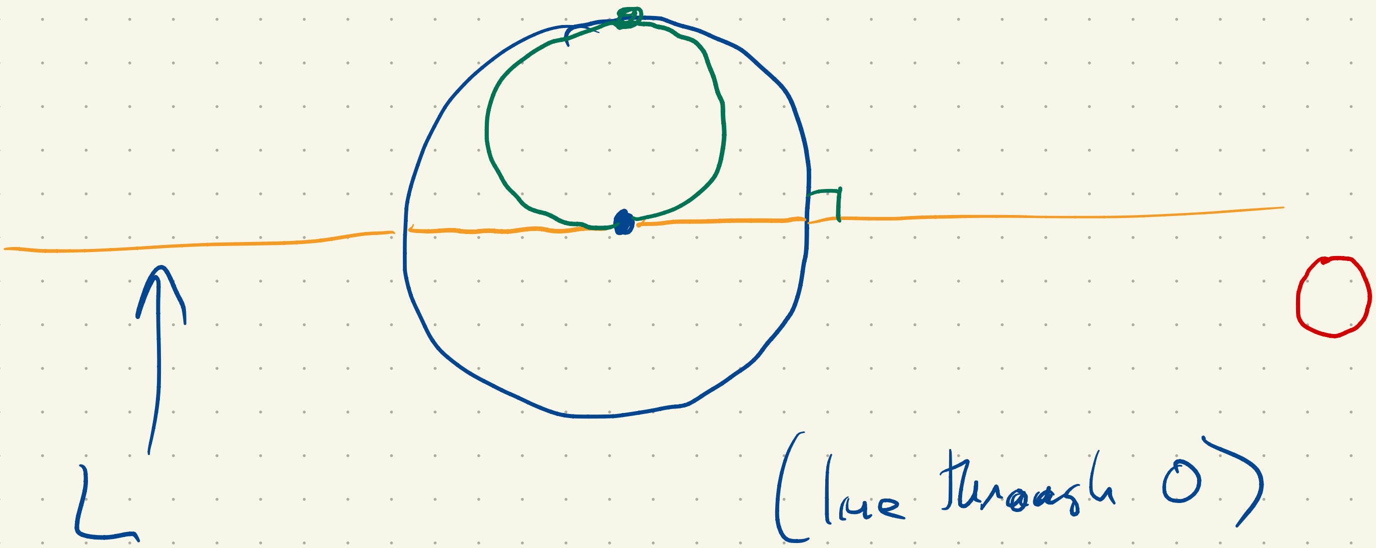
d) Show if $|Tz| < 1 \Rightarrow |z| < 1$.

(use a) and c))

e) conclude $T(D) = D$

What should a line be in this geometry?





(line through 0)

special features:

- meets S^1 at right angles
- is a Möbius line through $0^* = \infty$.

If T is a hyperbolic transformation

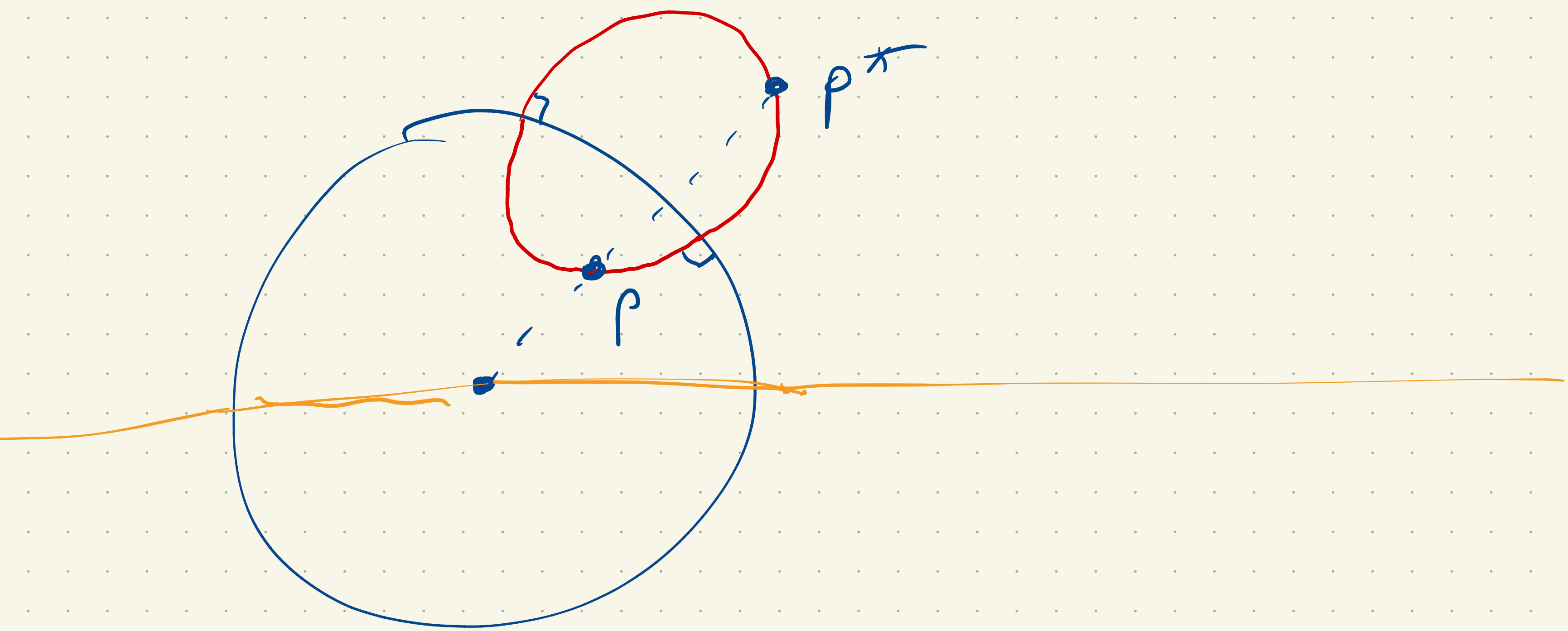
$T(L)$ is a Möbius line.

Let $p = T(0) \in D$

$$T(\infty) = T(0^*) = \left(T(0)\right)^*$$

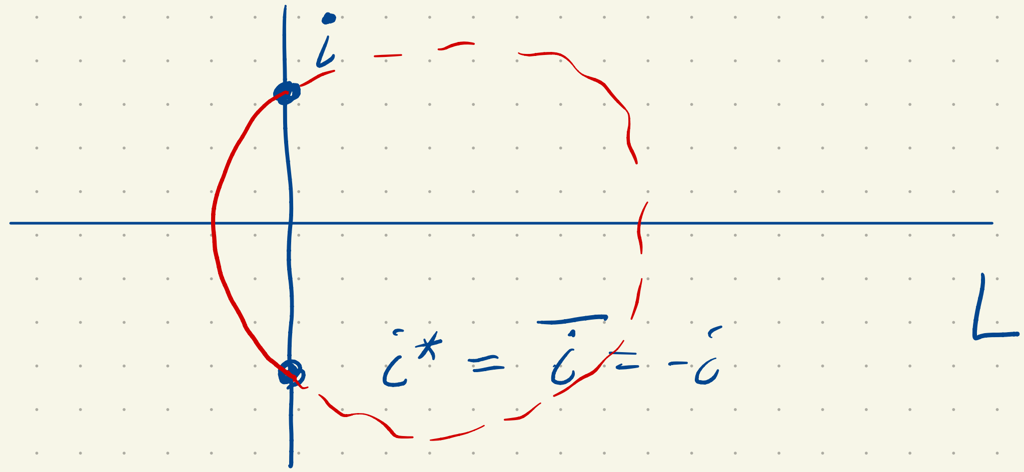
$$= p^*$$

$\infty \in L$ so $T(\infty) \in T(L)$.



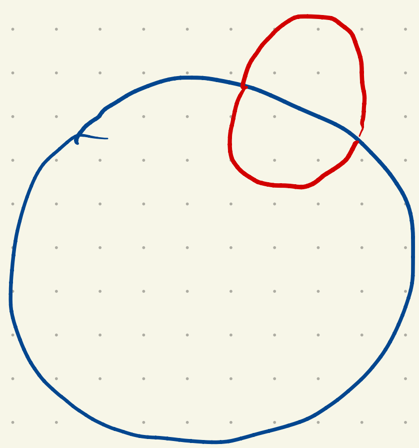
Def: A hyperbolic line is a Möbius line that intersects S^1 at right angles.

HW: Let L be a Möbius line and z a point not on L. If L' is a Möbius line through z then L' meets L at right angles if and only if z^* is on L' where z^* is the reflection of z about L.



Prop: Let $z \in \mathbb{D}$. A Möbius line L incident with z is a hyperbolic line if and only if $z^* \in L$.

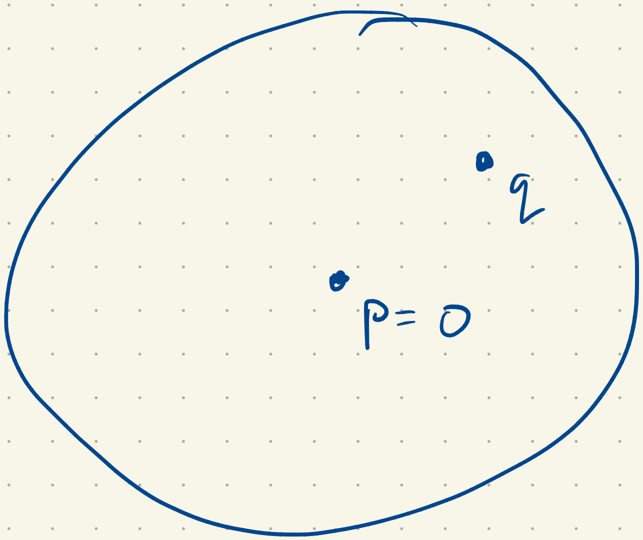
(mirror w.r.t. S^1)



Q: Given two points in $\mathbb{C}$ how many Möbius lines pass through them?

Goal: Given $p, q \in D$ $p \neq q$ there is a unique hyperbolic line incident to both.

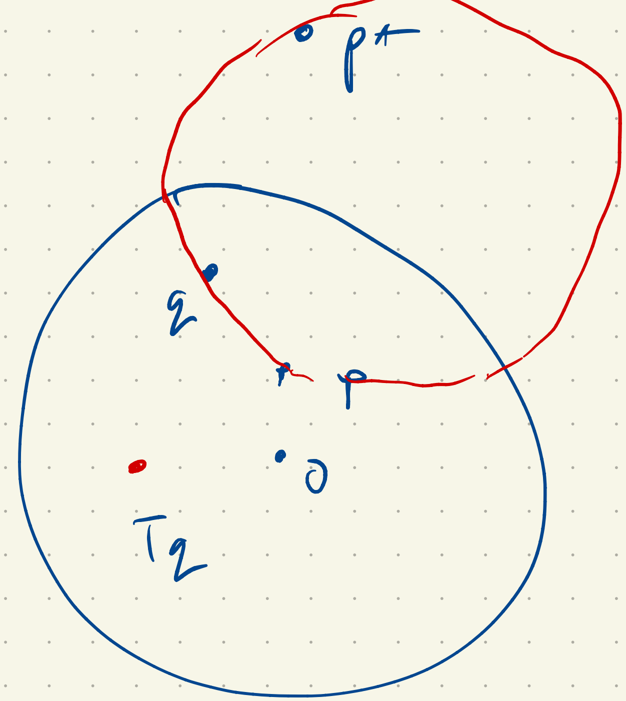
$p = 0$



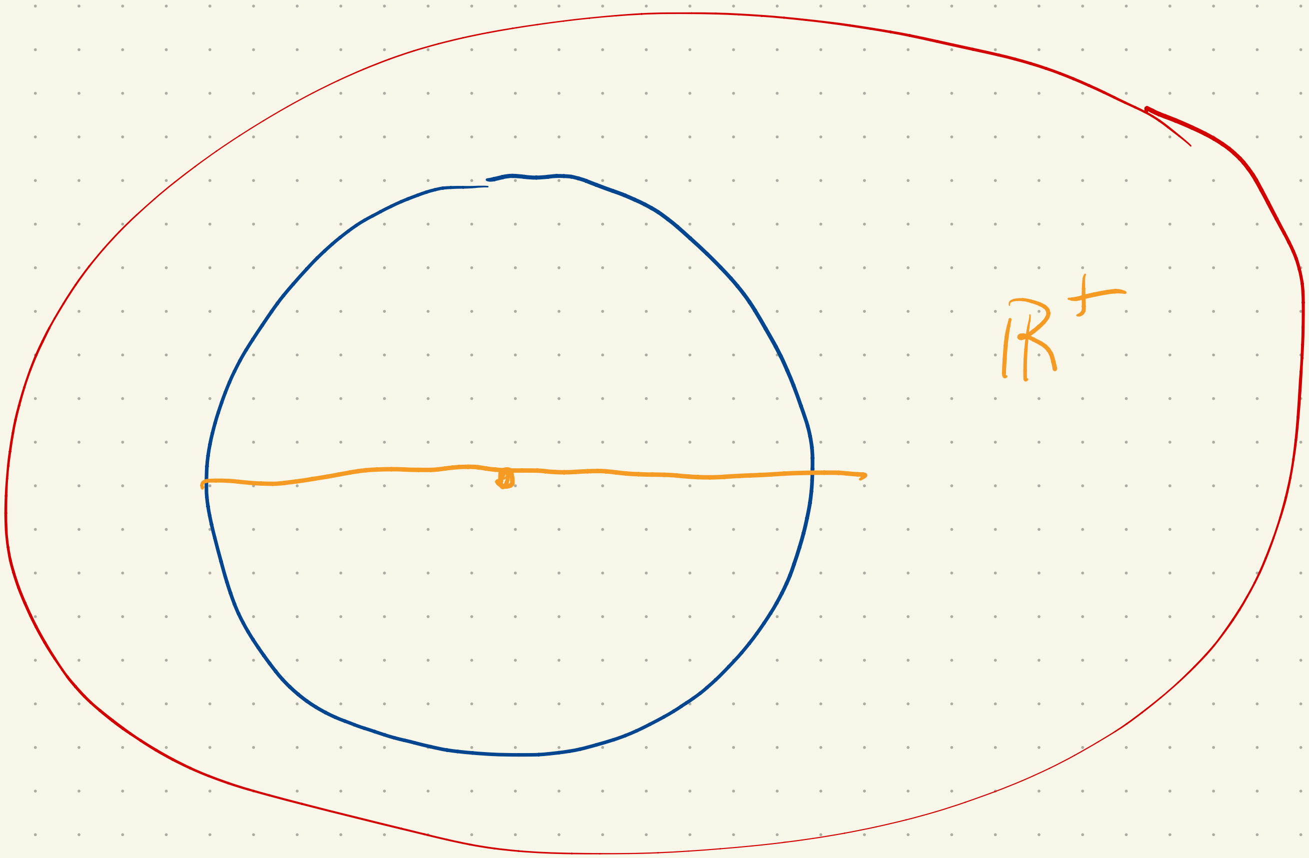
Easy if $p = 0$

$q \neq 0$

$$\frac{z - p}{1 - \bar{p} z}$$



p, q, p^* determine a unique Möbius line, and it's a hyperbolic line.



Every hyperbolic line is congruent to $\mathbb{R}^+$