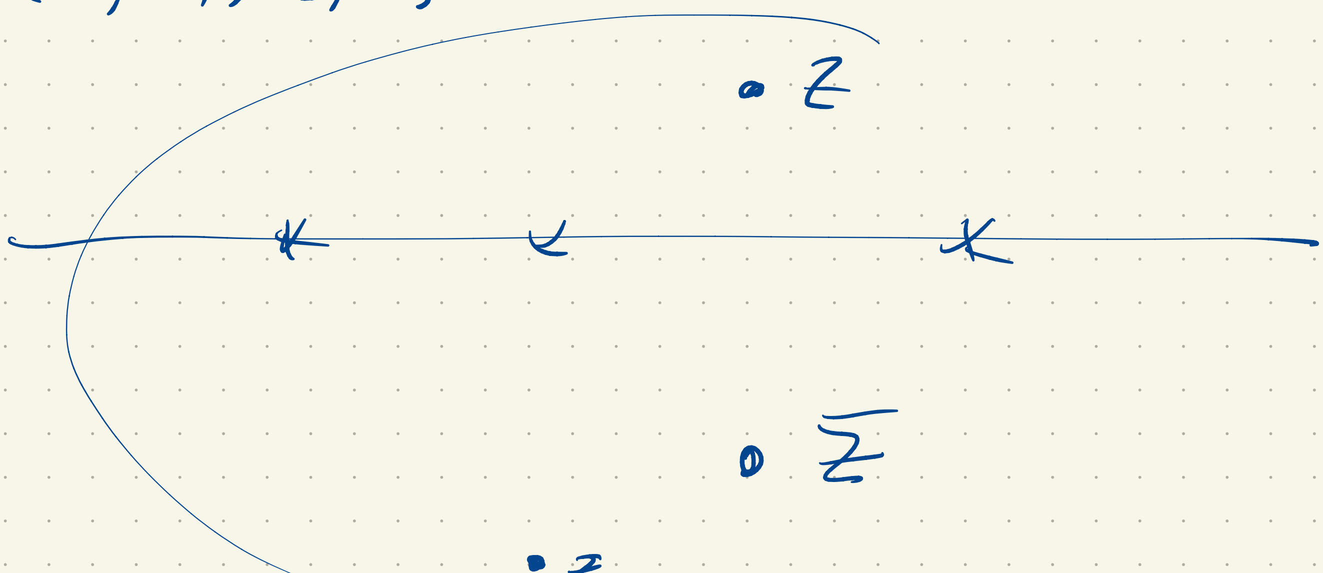
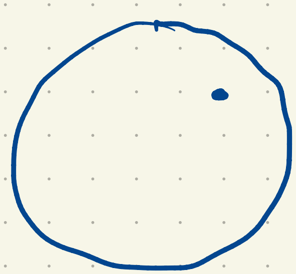
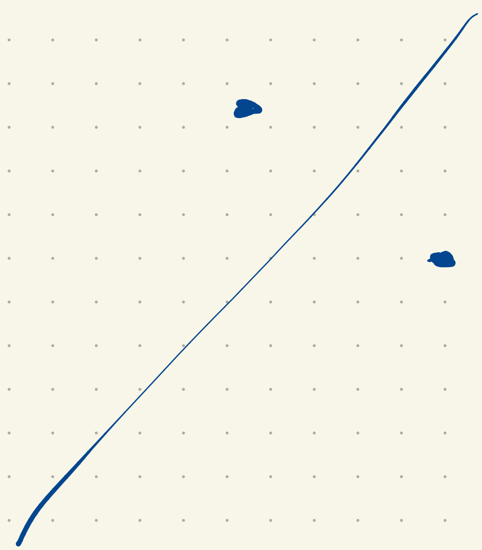
$$z \longmapsto z^*$$

$$z_1, \; z_2, \; z_3$$

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$







$z_1 = 1, \quad z_2 = i, \quad z_3 = -i$

z

$$(z, z_1, z_2, z_3)$$

$$(z, 1, i, -i) = \frac{z-i}{z+i} \frac{1+i}{1-i}$$

$$(z^*, 1, i, -i) = \frac{z^*-i}{z^*+i} \frac{1+i}{1-i}$$

$$(z^*, 1, i, -i) = \overline{(z, 1, i, -i)}$$

$$= \overline{\frac{z-i}{z+i} \frac{1+i}{1-i}}$$

$$= \frac{\bar{z}+i}{\bar{z}-i} \frac{1-i}{1+i}$$

$$= \frac{1 + i\bar{z}^{-1}}{1 - i\bar{z}^{-1}} \frac{i+1}{i-1}$$

$$= \frac{-i + \bar{z}^{-1}}{-i - \bar{z}^{-1}} \frac{i+1}{i-1}$$

$$= \boxed{\frac{\bar{z}^{-1} - i}{\bar{z}^{-1} + i} \frac{1+i}{1-i}}$$

$$= (\bar{z}^{-1}, 1, i, -i)$$

$$(z^*, 1, i, -i) = (\bar{z}^{-1}, 1, i, -i)$$

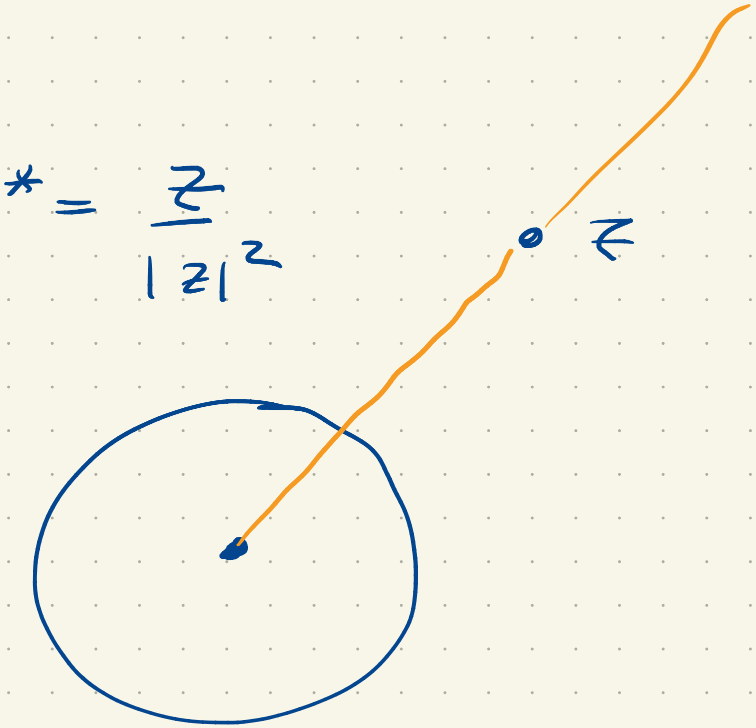
$$(a, 1, i, -i) = (b, 1, i, -i)$$

$$z^* = \bar{z}^{-1}$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$\bar{z}^{-1} = \frac{z}{|z|^2}$$

$$z^* = \frac{z}{|z|^2}$$



Exercise: For a circle of radius R centered at O

$$z^* = R^2 \frac{z}{|z|^2}$$

Steiner Circles

"What do Möbius transformations actually do?"

Fixed points:

a) T is the identity and everything is fixed

b) T has two fixed points

c) T has one fixed point

$$Tp = p \qquad Tq = q$$

$\cdot\, q$

$\cdot\, p$

Find a Möbius transformation S

$$S(p) = 0$$
$$S(q) = \infty$$

$$S(z) = \lambda_0 \frac{z - p}{z - q}$$

$$\begin{tikzcd}
\mathbb{C}^+ \arrow[r, "R"] \arrow[d, "S^{-1}"'] & \mathbb{C}^+ \arrow[d, dotted, bend right, "S^{-1}"'] \\
\mathbb{C}^+ \arrow[r, "T"'] & \mathbb{C}^+ \arrow[u, "S"']
\end{tikzcd}$$

$$S(p) = 0$$
$$S^{-1}(0) = p$$

$$R = S \circ T \circ S^{-1}$$

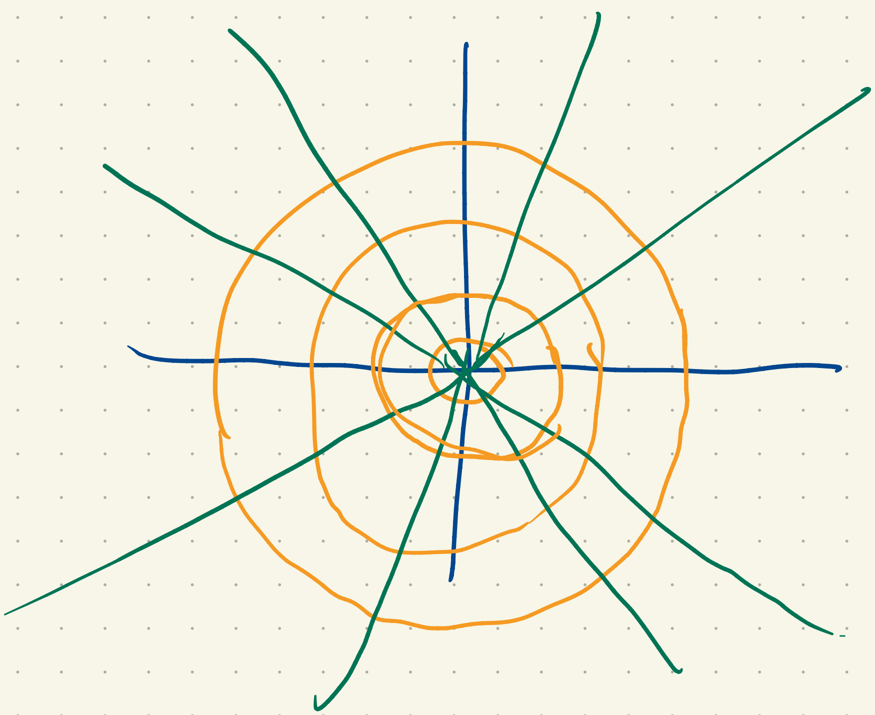
$$R(0) = 0$$
$$R(\infty) = \infty$$

$R(\infty) = \infty \Rightarrow R(z) = az + b \quad (a \neq 0)$

$R(0) = 0 \Rightarrow b = 0$

$R(z) = \cancel{a}z \quad (a \neq 0)$

$= \lambda z$



a) $\lambda = e^{i\theta} \quad \theta \in \mathbb{R}$

$(|\lambda| = 1)$

yellow circles go to themselves

green lines go to green lines

("elliptic")

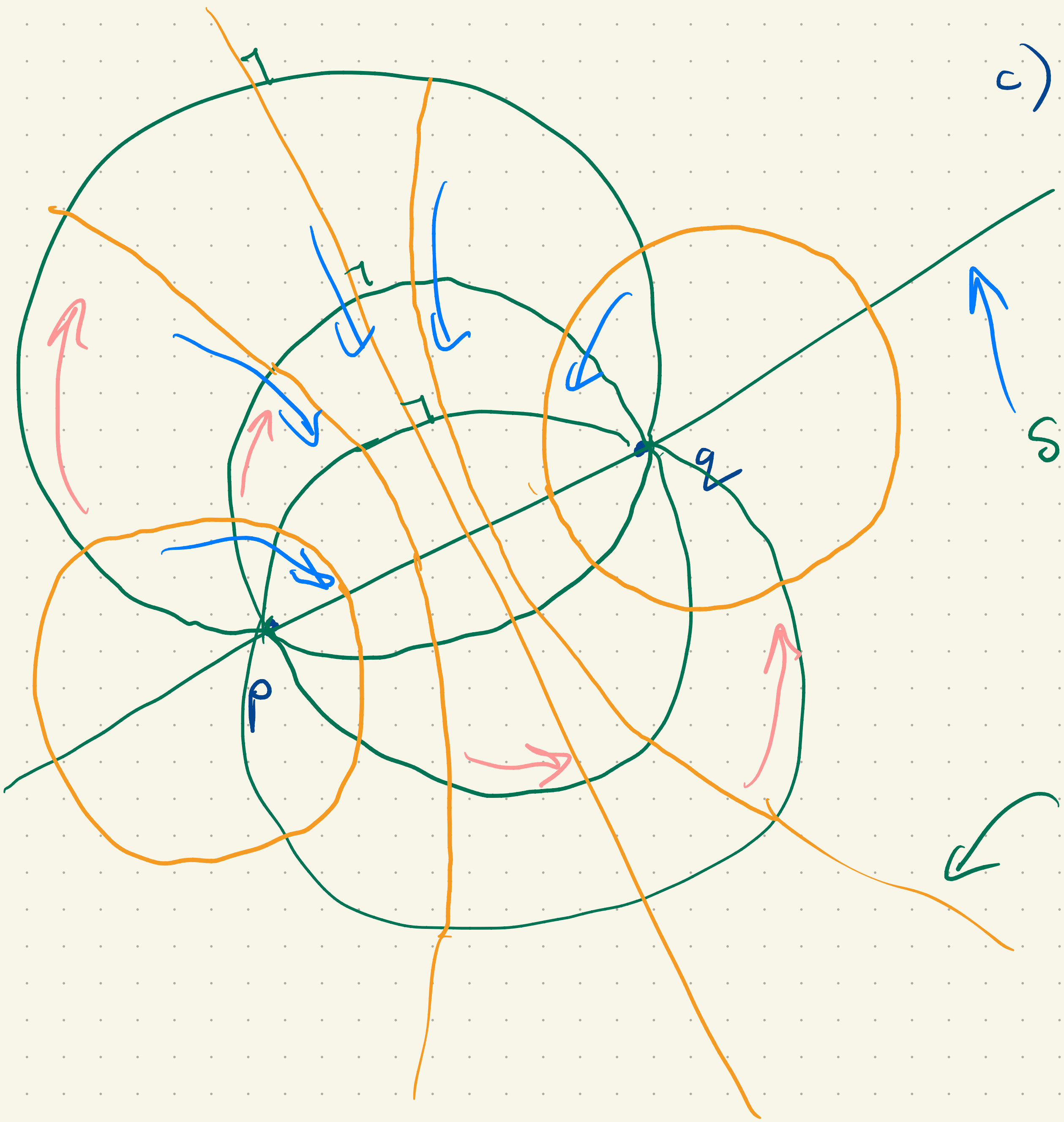
b) $\lambda \in \mathbb{R} \quad (\lambda \neq 0)$

green lines go to themselves

circles go to other circles

("hyperbolic")

c) None of the above
mix of above
"loxodromic"



Steiner circles of first kind.
All Möbius lines passing thru p and q.

Steiner circles of second kind,
perpendicular to
Steiner circles of first kind.

$$R = S \circ T \circ S^{-1}$$

$$S(z) = \lambda_0 \frac{z-p}{z-q}$$

$$R \circ S = S \circ T$$

$$R(z) = \lambda z$$

$$R \circ S(z) = \lambda \lambda_0 \frac{z-p}{z-q}$$

$$S \circ T(z) = \lambda_0 \frac{Tz-p}{Tz-q}$$

$$\lambda \lambda_0 \frac{z-p}{z-q} = \lambda_0 \frac{Tz-p}{Tz-q}$$

$$\frac{Tz-p}{Tz-q} = \lambda \frac{z-p}{z-q}$$

"Normal form of T"