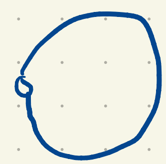
Last class

a) Cross ratio is an invariant of Möbius geometry

b) Four distinct points in $\mathbb{C}$ lie on a common line or circle iff their cross ratio is real.

Def. A Möbius line is a subset of $\mathbb{C}^+$ that is either a circle or is a straight line together with ∞.



Extensions:

b) is true replacing $\mathbb{C}$ with $\mathbb{C}^+$ if we also replace $\mathbb{R}$ with $\mathbb{R} \cup \{\infty\}$

Note: We really only need three distinct points (but there is no content with only 3)

Lemma: Given three distinct points in $\mathbb{C}^+$ there is a unique Möbius line that contains them.

Pf sketch: 1) No point is ∞.

a) colinear $\longrightarrow$ line, no circle

b) not colinear $\longrightarrow$ unique circle, no line.

2) One point is ∞ and two are not.

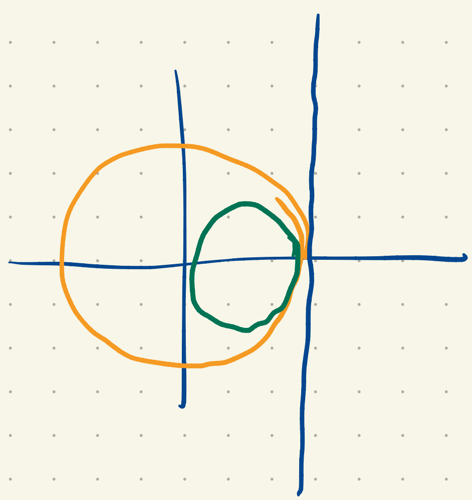
Easy.

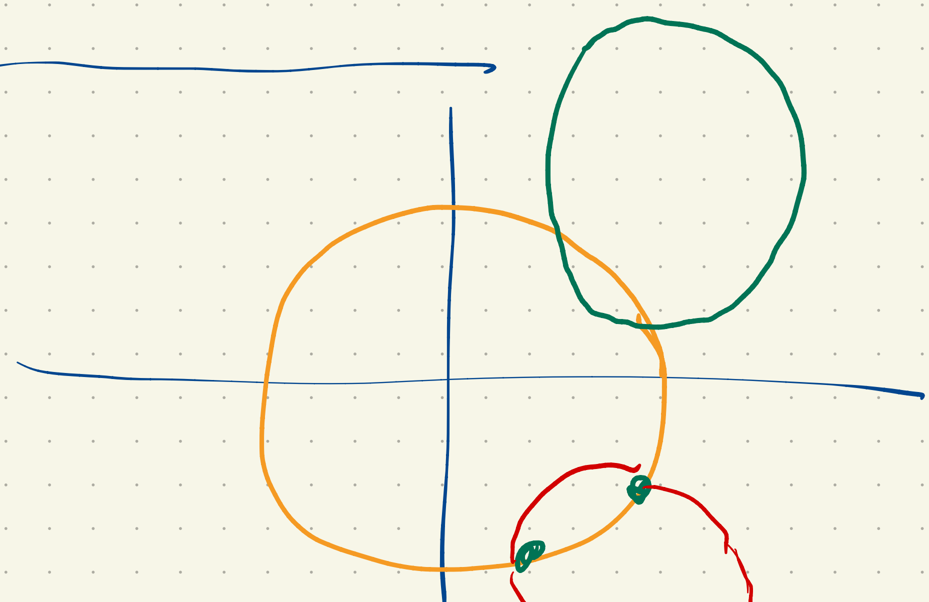
Thm: The image of a Möbius line under a Möbius transformation is a Möbius line.

Loosely: circle $\to$ either a circle or a line

line $\to$

$z \to \frac{1}{z}$





Pf sketch: Start with a Möbius line C.

Pick three distinct points on it, z_i $i=1,2,3$.

Let T be a Möbius transformation and ~~let $C' = T(C)$ and~~ let $w_i = Tz_i$ and let C' be the unique Möbius line containing the w_i's. Job: $C' = T(C)$.

Given $z \in \mathbb{C}^+$

$$z \in C \iff (z, z_1, z_2, z_3) \in \mathbb{R}$$

w_i's

$$\iff (Tz, Tz_1, Tz_2, Tz_3) \in \mathbb{R}$$

$$\iff (Tz, w_1, w_2, w_3)$$

$$\iff Tz \in C'.$$

Hence $C' = T(C)$. □

$$\underset{}{T}(\underset{\text{set}}{A}) = \underbrace{\{ Ta : a \in A \}}_{\text{image under } T \text{ of } A.}$$

Mirror symmetry.

Provisionally: Given distinct $z_1, z_2, z_3 \in \mathbb{C}^+$ let $z \in \mathbb{C}^+$.

We say z^* is the reflection of z about the z_i's if

$$z \longmapsto (z, z_1, z_2, z_3)$$

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

$ad - bc \neq 0$

$$z \longmapsto \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$

$$S(w) = (w, z_1, z_2, z_3)$$

$$S(w) = q \qquad q \in \mathbb{C}^+$$

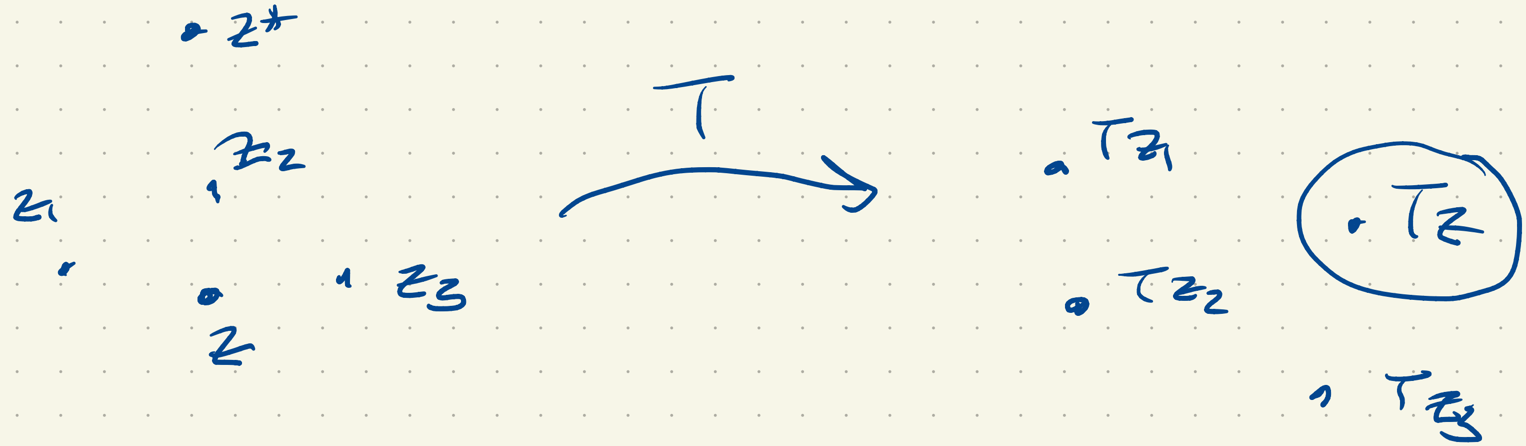
$$w = S^{-1}(q)$$

Given a Möbius transformation T

Claim z^* is the reflection of z about z_1, z_2, z_3

if and only if

Tz^* is ~~the~~ reflection of Tz about Tz_1, Tz_2, Tz_3.



Pf: $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$

$$\Updownarrow$$

$$(Tz^*, z_1, z_2, z_3) = \overline{(Tz, Tz_1, Tz_2, Tz_3)}$$

and hence Tz^* is $(Tz)^*$ iff z^* is the reflection of z about the z_i's.

$Tz \quad Tz_1 \quad Tz_2 \quad Tz_3$

$$\left((Tz)^*, Tz_1, Tz_2, Tz_3\right) = \overline{\left(Tz, Tz_1, Tz_2, Tz_3\right)}$$

Claim: If $z_1, z_2, z_3 \in \mathbb{R}$ and $z \in \mathbb{C}$ then

$$z^* = \overline{z}.$$

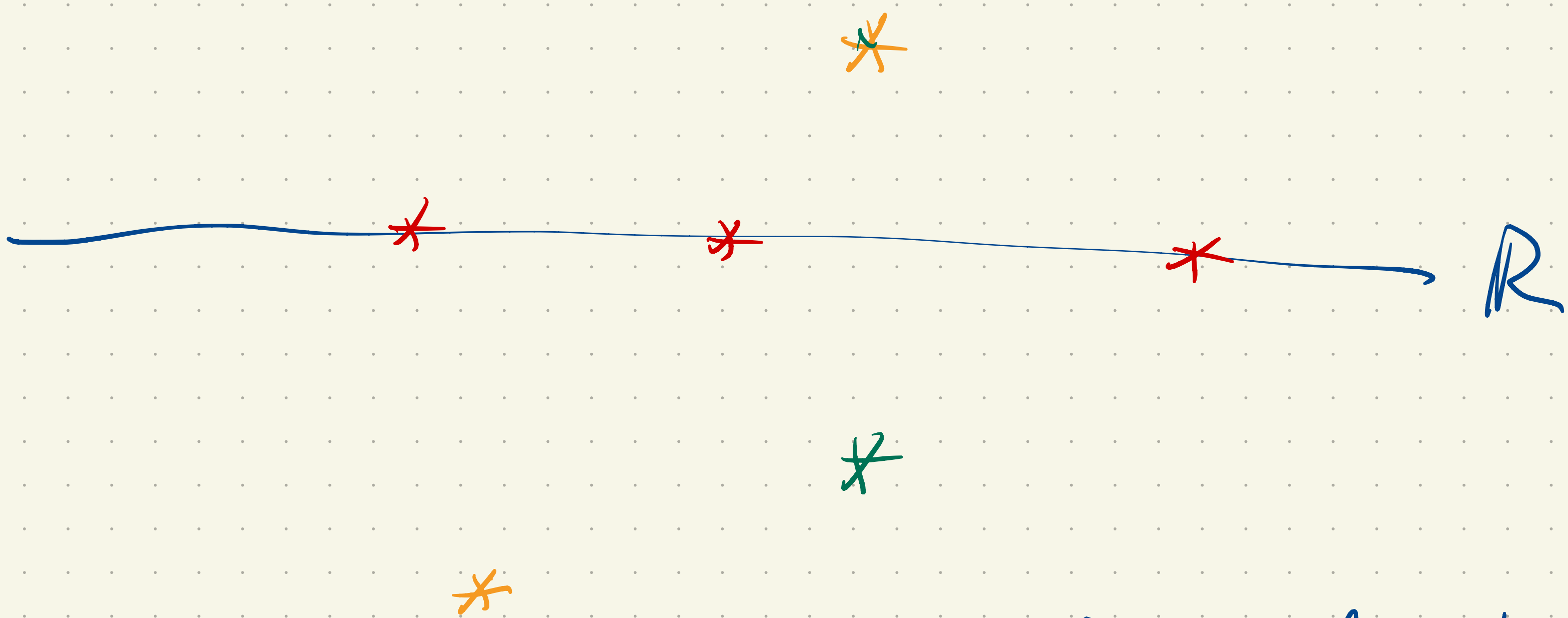
$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

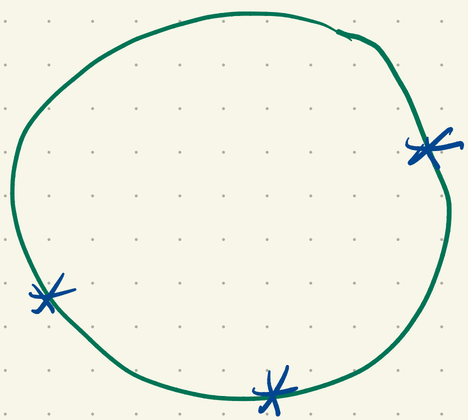
$$= \overline{\frac{(z - z_2)}{(z - z_3)} \frac{(z_1 - z_3)}{(z_1 - z_2)}}$$

$$= \frac{(\bar{z} - z_2)}{(\bar{z} - z_3)} \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

$$= (\bar{z}, z_1, z_2, z_3).$$

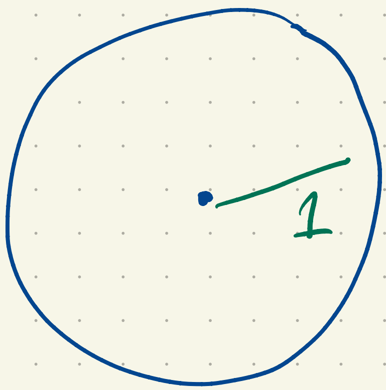
So $z^* = \bar{z}$.





Mirror symmetry depends only on the Möbius line determined by the z_i's.

$* \; z$

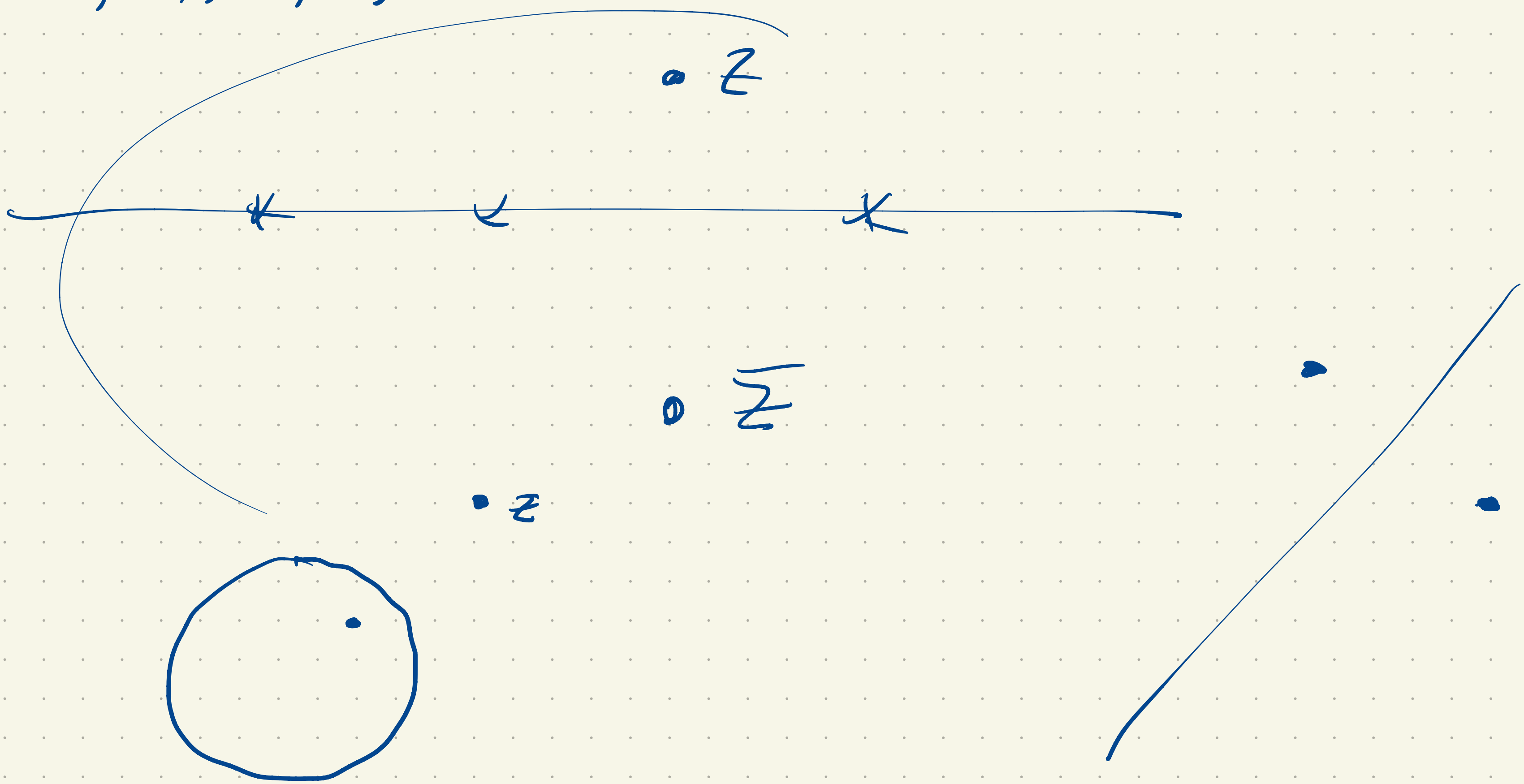


$$z^* = \frac{z}{|z|^2}$$

$$z \longmapsto z^*$$

$$z_1, \quad z_2, \quad z_3$$

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$



$z_1 = 1, \quad z_2 = i, \quad z_3 = -i$

z

$$(z, z_1, z_2, z_3)$$

$$(z, 1, i, -i) = \frac{z-i}{z+i} \frac{1+i}{1-i}$$

$$(z^*, 1, i, -i) = \frac{z^*-i}{z^*+i} \frac{1+i}{1-i}$$

$$(z^*, 1, i, -i) = \overline{(z, 1, i, -i)}$$

$$= \overline{\frac{z-i}{z+i} \frac{1+i}{1-i}}$$

$$= \frac{\bar{z}+i}{\bar{z}-i} \frac{1-i}{1+i}$$

$$= \frac{1 + i\bar{z}^{-1}}{1 - i\bar{z}^{-1}} \frac{i+1}{i-1}$$

$$= \frac{-i + \bar{z}^{-1}}{-i - \bar{z}^{-1}} \frac{i+1}{i-1}$$

$$= \boxed{\frac{\bar{z}^{-1} - i}{\bar{z}^{-1} + i} \frac{1+i}{1-i}}$$

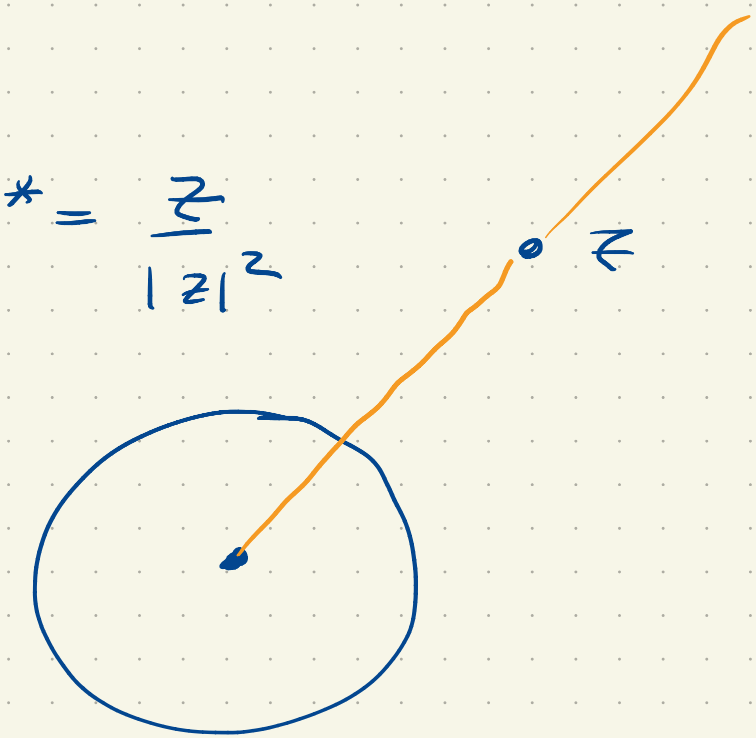
$$= (\bar{z}^{-1}, 1, i, -i)$$

$$(z^*, 1, i, -i) = (\bar{z}^{-1}, 1, i, -i)$$

$$(a, 1, i, -i) = (b, 1, i, -i)$$

$$z^* = \bar{z}^{-1}$$

$$z^* = \frac{z}{|z|^2}$$



$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

$$\bar{z}^{-1} = \frac{z}{|z|^2}$$

Exercise: For a circle of radius R centered at O

$$z^* = R^2 \frac{z}{|z|^2}$$