Last Class:

Fundamental Theorem of Möbius Geometry

Let z_1, z_2, z_3 and w_1, w_2, w_3 be two sets of distinct points in $\mathbb{C}^+$. There exists a unique Möbius transformation T with $T(z_i) = w_i$ $i = 1, 2, 3$.

$$T(z) = \frac{z - z_2}{z - z_3} \, \frac{z_1 - z_3}{z_1 - z_2}$$

$$T(z_1) = 1$$

$$T(z_2) = 0$$

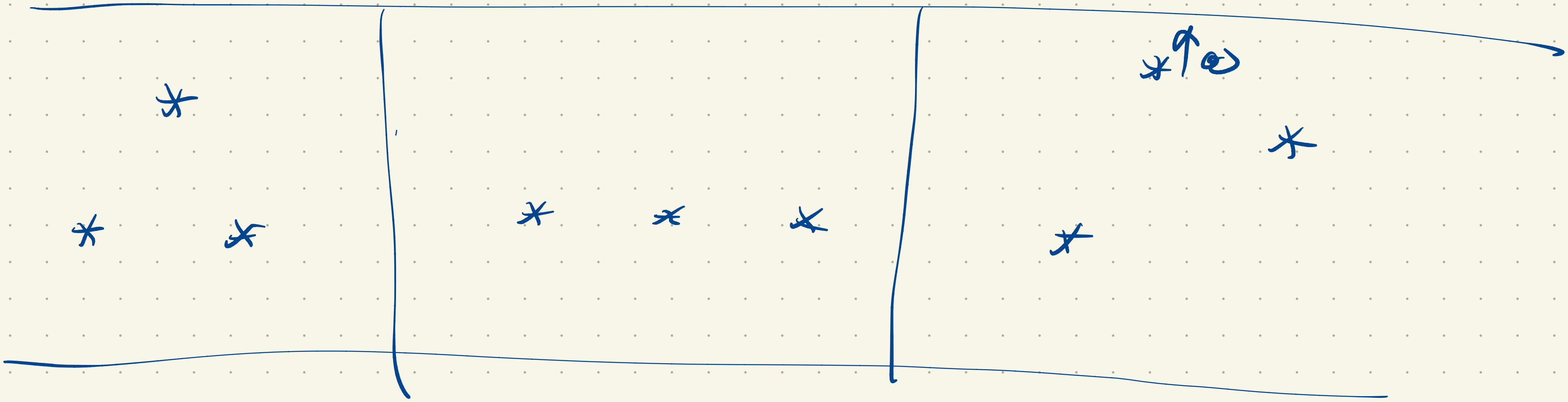
$$T(z_3) = \infty$$

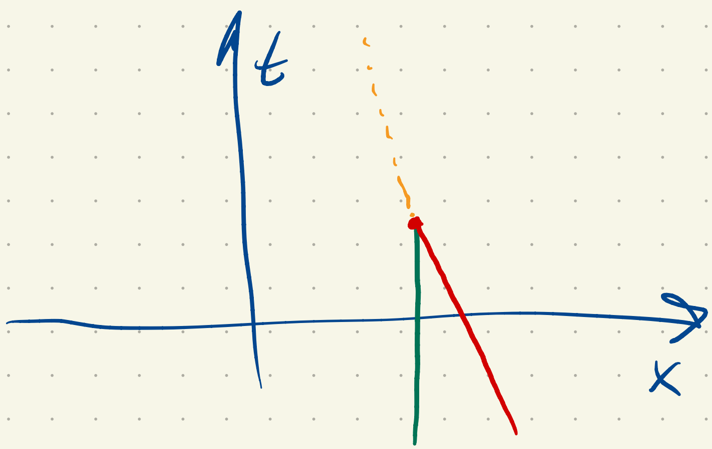
cross ratio

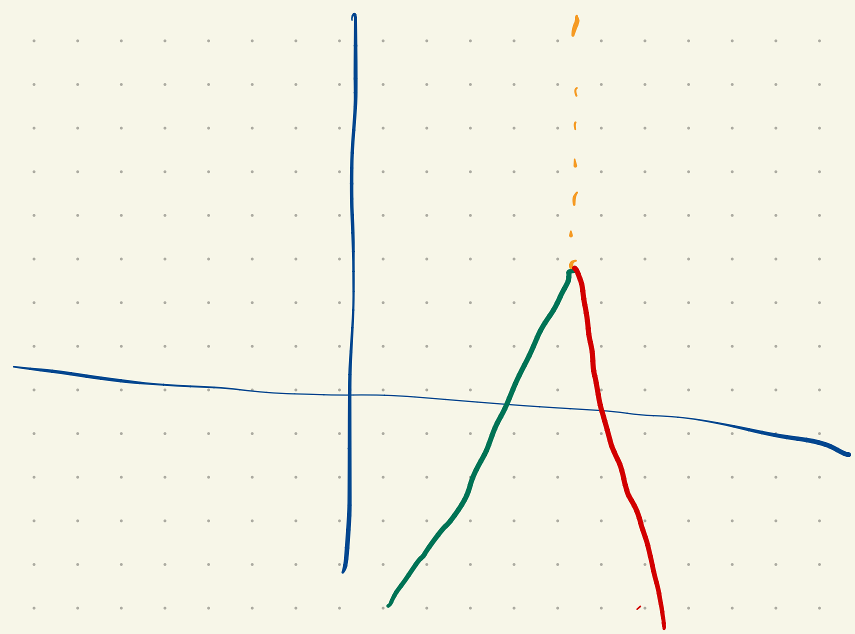
$$(z_0, z_1, z_2, z_3) = Tz_0$$

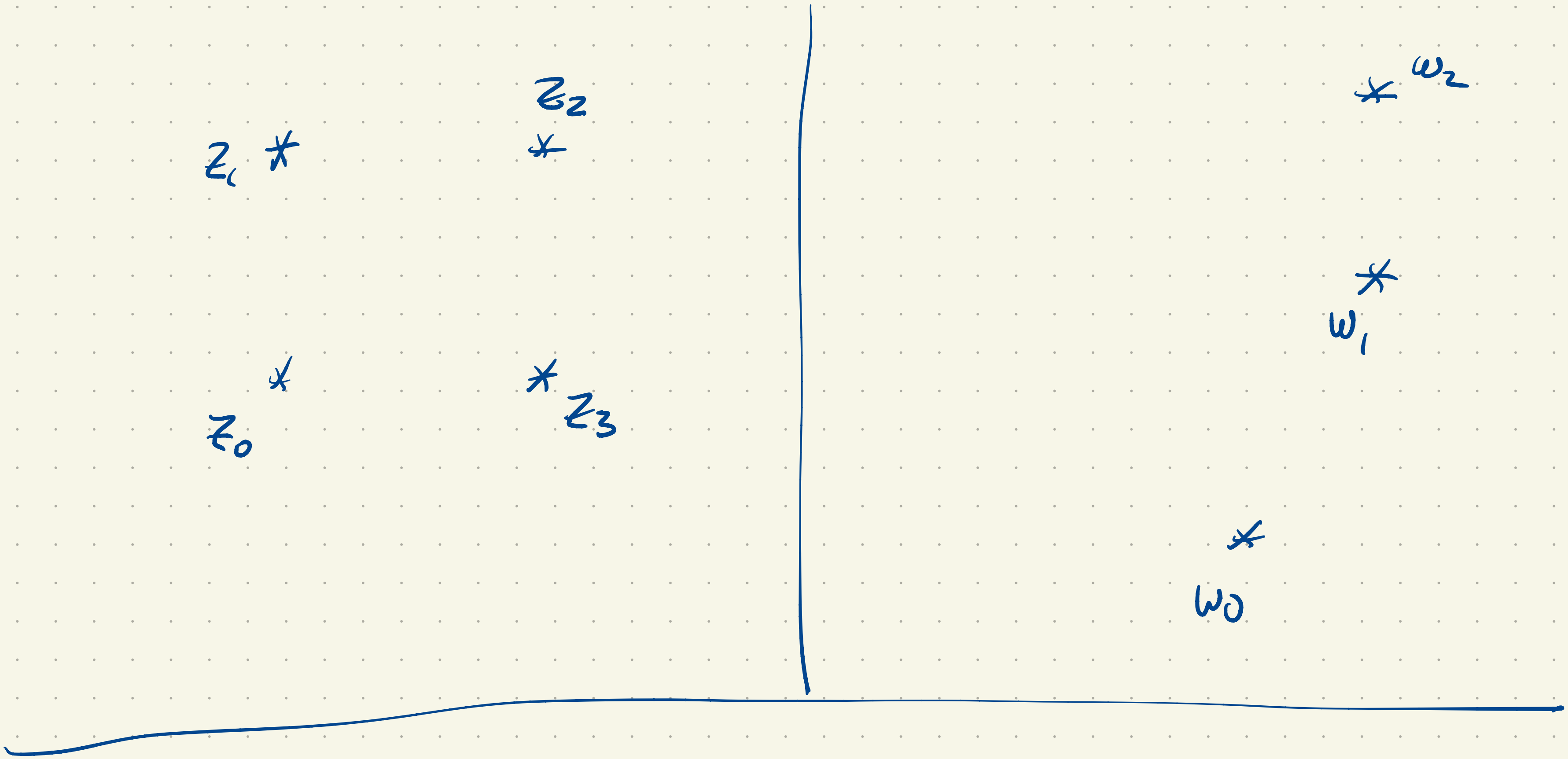
as in

All three point figures are congruent









Thm: The cross ratio of four distinct points in $\mathbb{C}^+$ is invariant under Möbius transformations.

Pf: Fix four distinct points $z_0, z_1, z_2, z_3 \in \mathbb{C}^+$.

Let T be an arbitrary Möbius transformation.

Define $S(z) = (z, z_1, z_2, z_3)$.

Define $R(z) = (Tz, Tz_1, Tz_2, Tz_3)$.

Observe S and R are both Möbius transformations.

$$\left[w \longmapsto (w, Tz_1, Tz_2, Tz_3) \right]$$

Note $S(z_1) = 1$ $\quad (w_1, w_1, w_2, w_3) = 1$

$S(z_2) = 0$ $\quad (z_1, z_1, z_2, z_3) = 1$

$S(z_3) = \infty$.

But $R(z_1) = (Tz_1, Tz_1, Tz_2, Tz_3)$

$= 1,$

$$R(z_2) = 0$$

$$R(z_3) = \infty,$$

Hence $S = R$ and therefore for any z_0

$$(z_0, z_1, z_2, z_3) = (Tz_0, Tz_1, Tz_2, Tz_3).$$

$\square$

By the way: angles are invariant also (HW).

What are cross ratios good for?

$$\frac{z-z_2}{z-z_3}\,\frac{z_1-z_3}{z_1-z_2}$$

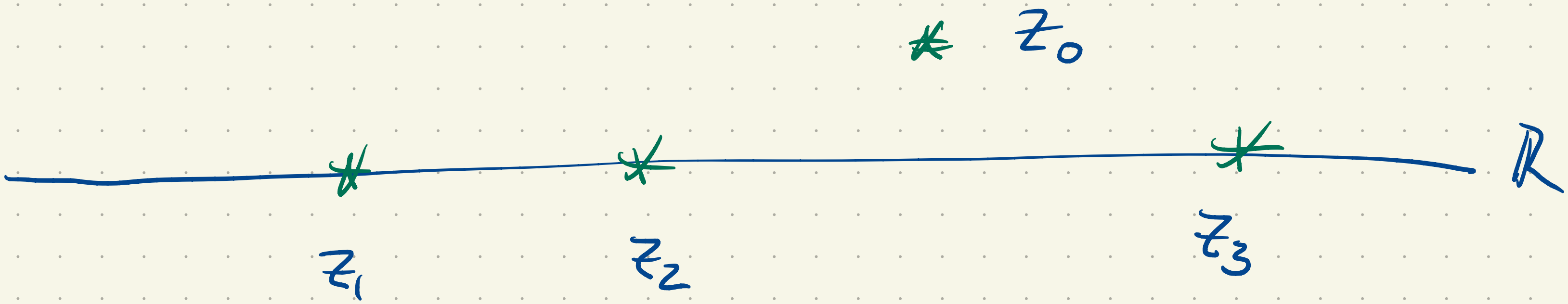
$$\underbrace{(z, 1, 0, \infty) = z}$$

all complex values are possible.

$$d(z, z_1)$$

z_1

When is the cross ratio real?



$$(z_0, z_1, z_2, z_3) = \underbrace{\frac{z_0 - z_2}{z_0 - z_3} \; \frac{z_1 - z_3}{z_1 - z_2}} \overset{?}{\in} \mathbb{R}$$

$$\frac{z_0 - z_2}{z_0 - z_3} \in \mathbb{R}$$

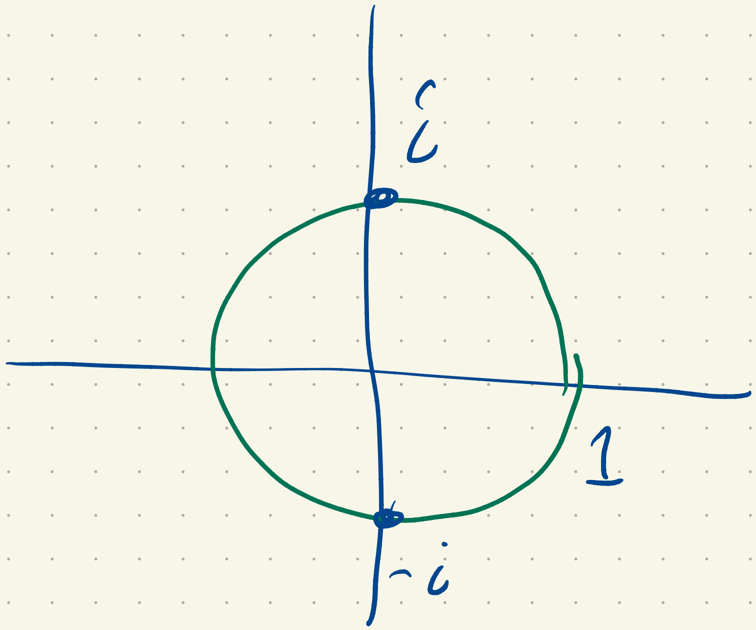
$$\alpha \in \mathbb{R}$$

$$z_0 - z_2 = \alpha(z_0 - z_3)$$

If z_0 is not real then

$$\frac{z_0 - z_2}{z_0 - z_3} \text{ is not real.}$$

$$(z,\ 1,\ i,\ -i) \quad \frac{z-i}{z-(-i)}\,\frac{1-(-i)}{1-i} = \frac{z-i}{z+i}\,\frac{1+i}{1-i}$$



$$= \frac{(\bar z - i)}{(\bar z - i)}\,\frac{z-i}{z+i}\,\frac{1+i}{1-i}\,\frac{1+i}{1+i}$$

$$1+2i-1$$

$$= \frac{-i\bar z - iz + |z|^2 - 1}{|z+i|^2}\,\frac{(1+i)^2}{2}$$

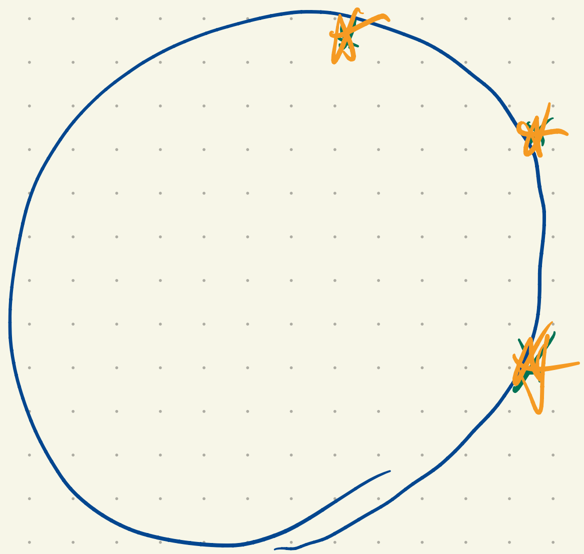
$$= \frac{-i(z+\bar z) + (|z|^2-1)}{|z+i|^2} \cdot \frac{2i}{2}$$

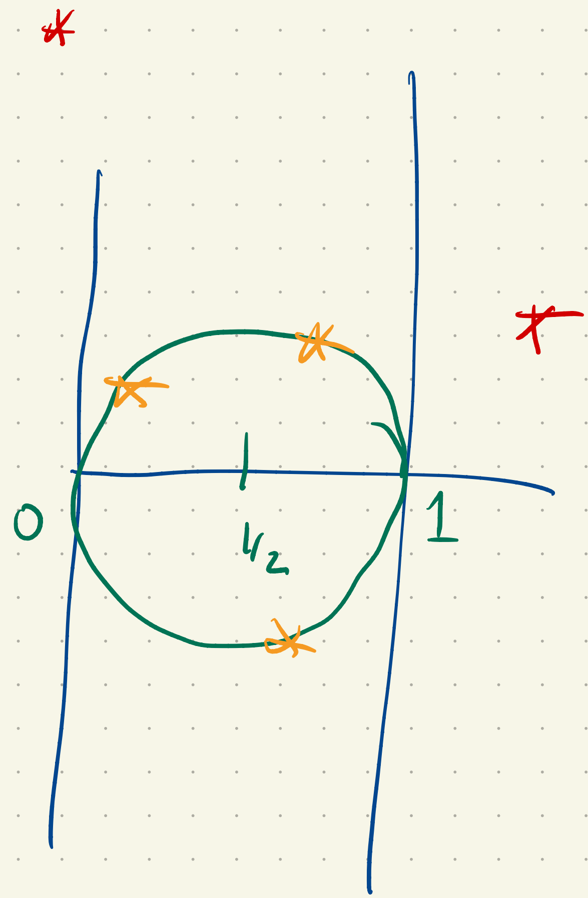
$$= \frac{(z+\bar z)}{|z+i|^2} + \frac{i}{|z+i|^2}\left(|z|^2-1\right)$$

Claim: Given $z_i \in \mathbb{C}$ $i = 0 \dots 3$, distinct

$$(z_0, z_1, z_2, z_3) \in \mathbb{R} \iff \text{the } z_i\text{'s lie on a common circle or line.}$$

Three non colinear points lie on a unique circle.





$$z \to \frac{1}{z}$$