$$Tz = \frac{az+b}{cz+d} \qquad T \longrightarrow M_T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

invertible

Claim $\quad Sz = \dfrac{ez+f}{gz+h}$

$(T \circ S)$ is a Möbius transformation and

$$M_{T\circ S} = M_T M_S$$

$$Tz = \frac{az+b}{cz+d} \qquad Sz = \frac{ez+f}{gz+h}$$

$$T(S(z)) = \frac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d}$$

$$= \frac{a(ez+f)+b(gz+h)}{c(ez+f)+d(gz+h)}$$

$$= \frac{(ae+bg)z+af+bh}{(ce+dg)z+cf+dh}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$M_T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M_{T \circ S} = M_T M_S$$

$$\det(M_{T \circ S}) = \det(M_T M_S)$$

$$= \det(M_T) \det(M_S) \neq 0$$

$$T \qquad (M_T)^{-1} \longrightarrow$$

$$\frac{1z + 0}{0z + 1}$$

$$(M_T)^{-1} \cdot M_T = I \rightarrow id$$

$$\uparrow \quad ad - bc \neq 0$$

Upshot: Every Möbius transformation is invertible $M_{T^{-1}} = (M_T)^{-1}$

$(\mathbb{C}^+, \mathcal{M})$ Möbius Geometry.

$\hookrightarrow$ Möbius transformations

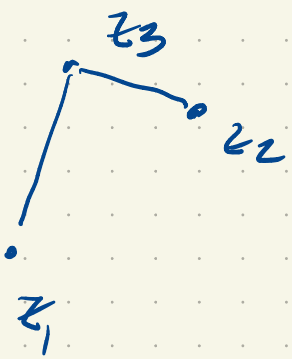
$$f(z) = e^{i\theta} z + b$$

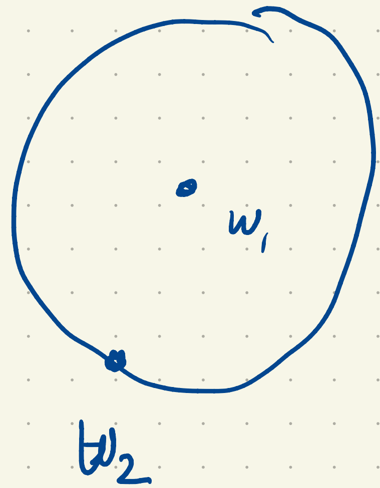
$$Tz = \frac{az+b}{cz+d}$$

$$M_T = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}.$$

$$\det(M_T) = \lambda^2 (ad - bc)$$

T Euclidean transformation.





Looks like T might be uniquely determined by its action on any 3 non colinear distinct points.

Claim: $z_1, z_2, z_3 \in \mathbb{C}$ There is a unique Möbius transformation

$w_1, w_2, w_3 \in \mathbb{C}$

$$z_i \to w_i$$

$i = 1, 2, 3.$

Def: A fixed point of a Möbius transformation

T is a point $z \in \mathbb{C}^+$ with $Tz = z$.

Every Möbius transf that is not the identity has at most 2 fixed points.

Claim Möbius transformations have at most 2 fixed points.

(except for the identity)

T, S

$$T(z_1) = w_1 = S(z_1)$$
$$T(z_2) = w_2 = S(z_2)$$
$$T(z_3) = w_3 = S(z_3)$$

$$\underbrace{(S^{-1} \circ T)}_{\mathrm{Id}}(z_i) = S^{-1}(w_i) = z_i \qquad i = 1, 2, 3$$

$$S^{-1} \circ T = Id$$

$$T = S$$

$$Tz = z \qquad \infty \longmapsto \frac{a}{c}$$

$$\frac{az+b}{cz+d} = z$$

$$z^2 = 0$$

$$az + b = z(cz+d)$$

$$z^2 = a$$

$$cz^2 + dz - az - b = 0$$

$$cz^2 + (d-a)z - b = 0$$

$c \neq 0$ at most two roots in $\mathbb{C}$ and ∞ is not a fixed point.

at most two fixed points

$c \neq 0 \Rightarrow d \neq 0$

$c = 0$ $\qquad \dfrac{az+b}{d}$

$ad - bc$

$Tz \quad az+b \qquad a \neq 0$

$T\infty = \infty \qquad az+b = z$

$(a-1)z = -b$

$z = \dfrac{-b}{a-1}$ (if $a \neq 1$)

$z + b = z \qquad b \neq 0$ only ∞ is a f.p.

$b = 0$ all points are.

$T(\infty) = \frac{a}{c}$ $\quad a \neq 0 \quad c = 0 \quad \frac{a}{c} = \infty$

$z_1 \to 1$ $\quad w_1 \to 1$

$z_2 \to 0$ $\quad w_2 \to 6$

$z_3 \to \infty$ $\quad w_3 \to \infty$

$$Tz = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$ $\quad z_1, z_2, z_3 \in \mathbb{C}$

$\infty \to 1$

$z_2 \to 0$

$z_3 \to \infty$

$$Tz = \frac{z - z_2}{z - z_3}$$

$z_1 \to 1$

$\infty \to 0$

$z_3 \to \infty$

$$\frac{z_1 - z_3}{z - z_3}$$

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\underbrace{(z_0, z_1, z_2, z_3)}_{\text{cross ratio}} := \frac{(z_0 - z_2)(z_1 - z_3)}{(z_0 - z_3)(z_1 - z_2)}$$