$$\mathrm{Area}(\{z_0, z_1, z_2\}) = \frac{1}{2}\left|\mathrm{Im}\left((z_1-z_0)\overline{(z_2-z_0)}\right)\right|$$

$A_i \subseteq S$

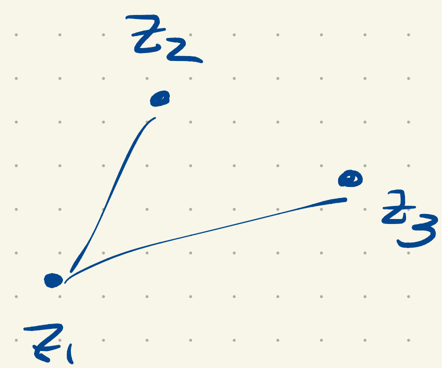
$$T((A_1, A_2, A_3)) := (TA_1, TA_2, TA_3)$$

$C \leftarrow$ collection of tuples of figures

(z_1, z_2)

(z_1, z_2, z_3)

$(\{z_1\}, \{z_2\}, \{z_3\})$



$\mathcal{C} \leftarrow$ invariant collection of tuples of figure

$$(A_1, \dots, A_k) \in \mathcal{C} \Rightarrow (TA_1, \dots, TA_k) \in \mathcal{C}$$

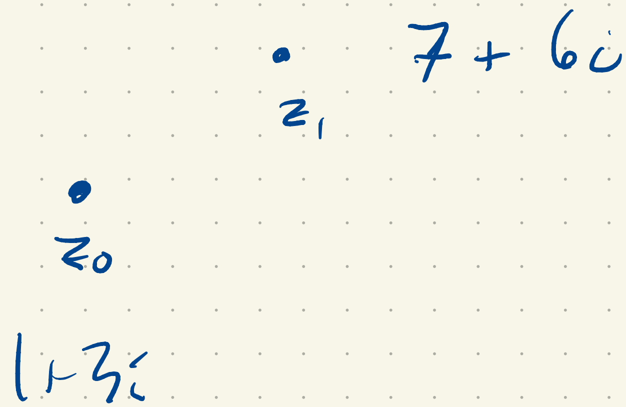
$$f : \mathcal{C} \to \, ?$$

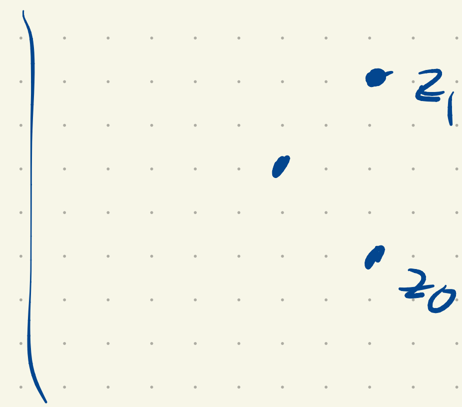
invariant if $\quad f(T((A_1, \dots, A_k))) = f((A_1, \dots, A_k))$

$$f(\text{———}) \to \theta$$

$$C = \{ (z_0, z_1) : z_i \in \mathbb{C} \}$$

$$f(z_0, z_1) = \begin{cases} 1 & \operatorname{Re}(z_1 - z_0) > 0 \\ 0 & \text{otherwise} \end{cases}$$





Claim: f is invariant under translation but not oriented Euclidean geometry.

$T \in \mathcal{G}_{trans} \longrightarrow Tz = z + b$

$$f((Tz_0, Tz_1)) = f((z_0, z_1))$$

Let $T \in \mathcal{G}_{trans}$.
Then there exists $b \in \mathbb{C}$ such that
$Tz = z + b$

$$\downarrow$$

$$\mathrm{Re}(Tz_1 - Tz_0) = \mathrm{Re}((z_1 + b) - (z_0 + b))$$
$$= \mathrm{Re}(z_1 - z_0)$$

$$f((Tz_0, Tz_1)) = \begin{cases} 1 & \mathrm{Re}(Tz_1 - Tz_0) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \mathrm{Re}(z_1 - z_0) \geq 0 \\ 0 & \text{---} \end{cases}$$

$$= f((z_0, z_1))$$

$Tz = -z$ $\qquad (i, 3)$

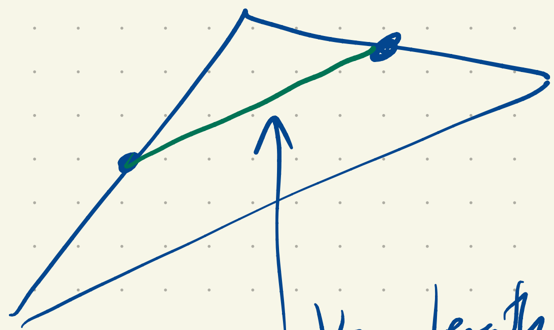
$$f((i,3)) = 1$$

$$f((Ti, T3)) = f((-i, -3)) = 0$$

$f(z_0, z_1, z_2)$

invariant under $\mathcal{G}_{0\,Euc}$

not under $\mathcal{G}_{Euc}$



1/2 length of other side (and is parallel)

Models of a geometry

Möbius Geometry

$$Tz = \frac{az+b}{cz+d}$$

$a, b, c, d \in \mathbb{C}$

$ad - bc \neq 0$

Möbius transformation

$T: \mathbb{C} \to \mathbb{C}$

We'll develop this as a map $\mathbb{C}^+ \to \mathbb{C}^+$

Issues:

1) division by 0. $\frac{a}{0} = \infty$ if $a \neq 0$

$\frac{0}{0}$ is undefined.

There is at most one z where $cz + d = 0$.

If $c \neq 0$ this occurs at $z = -d/c$

$\neq 0!$

$$T(-d/c) = \frac{a\left(-d/c\right) + b}{0} = \frac{-\frac{1}{c}\left[ad - bc\right]}{0}$$

$= \infty$

If $c = 0$ $\quad cz + d = d$

$ad - bc \neq 0$

so $d \neq 0$.

$cz + d \neq 0 \quad \forall z$

$ad - bc$

$$T(\infty) = \frac{a \cdot \infty + b}{c \cdot \infty + d} = \frac{a}{c}$$

$b = 1,\ d = 1$

$a = 3$

$c = 1$

$\rightarrow$ if $c \neq 0$

$$\frac{3z + 1}{z + 1}$$

$$T(z) = \frac{a}{d} z + \frac{b}{d}$$

$$T(\infty) = \frac{a}{c}$$

$$\lim_{z \to \infty} T(z) = \frac{a}{c} \quad \text{if } c \neq 0$$

$$\lim_{z \to \infty} T(z) = \infty \quad \text{if } c = 0$$

$$z \to \frac{1}{z} \qquad\qquad z + b$$

$$e^{i\theta} z + b$$

$$z \longmapsto z + b$$

$$z \longmapsto az \qquad a \neq 0$$

$$z \longmapsto 1/z$$