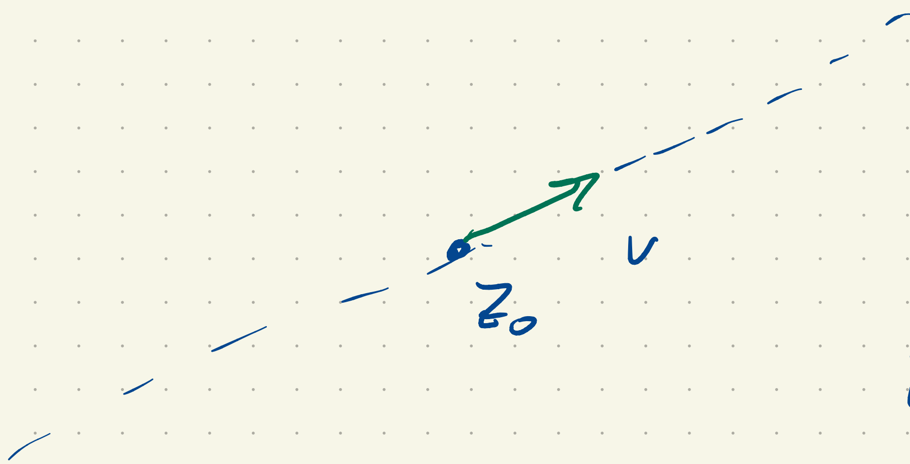
# Invariants

A $\underbrace{\text{figure}}_{\text{subset}}$ $A \subseteq \mathbb{C}$ is called a line if

there exists $z_0 \in \mathbb{C}$ and a $v \in \mathbb{C} \setminus \{0\}$

such that $A = \{ z_0 + tv : \underline{t \in \mathbb{R}} \}$

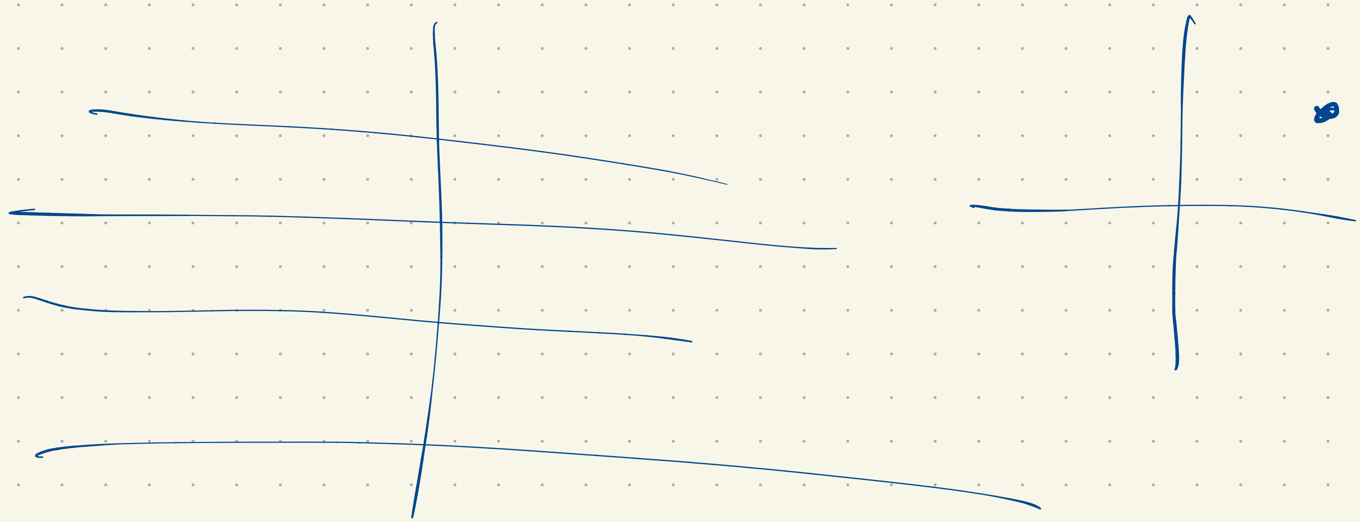



On HW: If $L$ is a line and $T$ is an oriented Euclidean transformation then $T(L)$ is a line

$\mathcal{L} = \{ \text{all lines in } \mathbb{C} \}$

$\forall L \in \mathcal{L}, \ \forall T \in \mathcal{G}_{OEuc}, \ T(L) \in \mathcal{L}$

such a collection is called an invariant collection of figures



$$\{ T(L) : T \in \mathcal{G}_{OEuc} \} = \mathcal{L}$$

$\mathcal{D}$ = set of all two point subsets of $\mathbb{C}$

$\{z_1, z_2\} \in \mathcal{D}$ $\quad z_1 \neq z_2$

Is $\mathcal{D}$ invariant under $\mathcal{G}_{0\,Euc}$

$$T(\{z_1, z_2\}) = \underbrace{\{Tz_1, Tz_2\}}_{\in \mathcal{D}}$$

T is injective since it is bijective (it has an inverse!)

$$\{T(\{0,1\}) : T \in \mathcal{G}_{0\,Euc}\} \neq \mathcal{D}$$

$$d : \mathcal{D} \longrightarrow \mathbb{R}$$

$$d(\{z_1, z_2\}) = |z_1 - z_2| \left( = |z_2 - z_1| \right)$$

Claim: $$d\left(\underline{T(\{z_1, z_2\})}\right) = d\left(\{z_1, z_2\}\right)$$

"invariant function"

$$T(z) = e^{i\theta} z + b$$

$$T(\{z_1, z_2\}) = \{e^{i\theta} z_1 + b,\ e^{i\theta} z_2 + b\}$$

$$d\left(T(\{z_1, z_2\})\right) = \left| (e^{i\theta} z_1 + b) - (e^{i\theta} z_2 + b) \right|$$

$$= \left| e^{i\theta} (z_1 - z_2) \right|$$

$$= |e^{i\theta}|\,|z_1 - z_2|$$

$$= |z_1 - z_2|$$

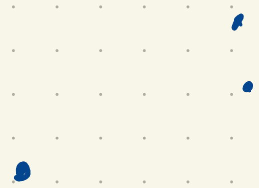
$$= d(\{z_1, z_2\}).$$

Def: Let $(S, \mathcal{G})$ be a geometry.

A collection $\mathcal{D}$ of figures is <u>invariant</u> if for all $A \in \mathcal{D}$ and all $T \in \mathcal{G}$, $T(A) \in \mathcal{D}$.

A function $f$ on an invariant collection $\mathcal{D}$ is itself <u>invariant</u> if $f(T(A)) = f(A)$ for all $A \in \mathcal{D}$ and all $T \in \mathcal{G}$.

E.g. $\mathcal{D}$ three point subsets of $\mathbb{C}$ (not colinear)



$\{z_1, z_2, z_3\}$

$$f(\{z_0, z_1, z_2\}) = \left|\mathrm{Im}\left((z_1 - z_0)\overline{(z_2 - z_0)}\right)\right|$$

Exercise: this is well defined (i.e.

$$\left|\mathrm{Im}\left((z_0 - z_1)\overline{(z_2 - z_1)}\right)\right| = \left|\mathrm{Im}\left((z_1 - z_0)\overline{(z_2 - z_0)}\right)\right|$$

$$T(z) = e^{i\theta} z + b$$

$$f\left(\{Tz_0, Tz_1, Tz_2\}\right) \stackrel{?}{=} f\left(\{z_0, z_1, z_2\}\right)$$

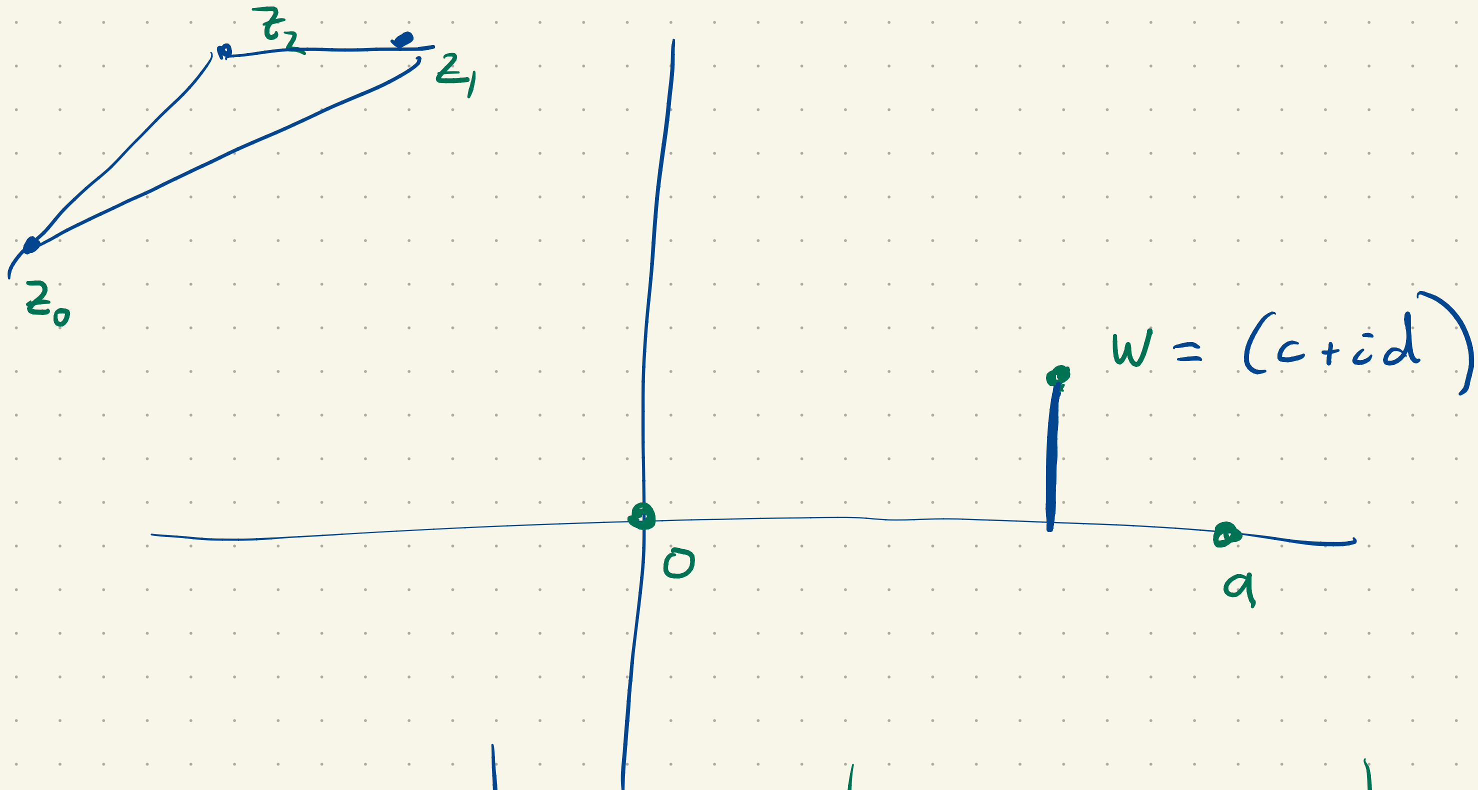
$$\left| \operatorname{Im}\left( e^{i\theta}(z_0 - z_1)\, \overline{e^{i\theta}(z_2 - z_1)} \right) \right|$$

$$\shortparallel$$

$$\left| \operatorname{Im}\left( e^{i\theta} e^{-i\theta} (z_0 - z_1)\, \overline{(z_2 - z_1)} \right) \right|$$

$$\shortparallel$$

$$f\left(\{z_0, z_1, z_2\}\right)$$




$$\left|\operatorname{Im}\left((z_1-z_0)\overline{(z_2-z_0)}\right)\right|$$

$$\left|\operatorname{Im}\left(a\overline{w}\right)\right|$$

$$a\left|\operatorname{Im}(\overline{w})\right| \qquad a\cdot d$$

$$a\left|\operatorname{Im}(w)\right|$$

$$\underbrace{\operatorname{Im}(c+id)}_{= d}, \qquad c,d\in\mathbb{R}$$

$$\text{Area}(\{z_0, z_1, z_2\}) = \frac{1}{2}\left|\operatorname{Im}\left((z_1-z_0)\overline{(z_2-z_0)}\right)\right|$$

$A_i \subseteq S$

$$T((A_1, A_2, A_3)) := (TA_1, TA_2, TA_3)$$

$\mathcal{C} \leftarrow$ collection of tuples of figures

$(z_1, z_2)$ $(z_1, z_2, z_3)$