What the heck is congruence?

These figures are "the same" from the point of view of Euclid

We'll have a set S of points. (1)

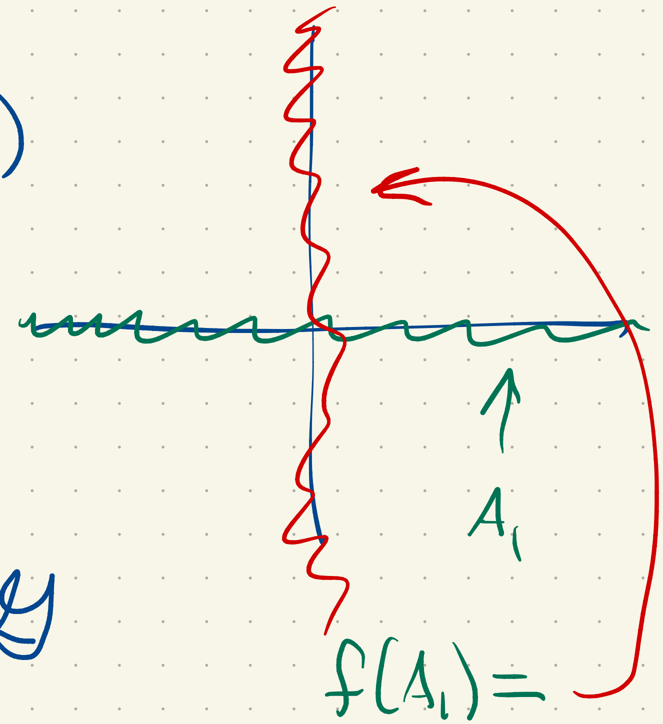
We'll have two subsets $A_1, A_2 \subseteq S$

We want to know if A_1 is $\underbrace{\text{congruent}}_{\cong}$ to A_2.

We encode the notion of congruence by means of preferred functions

$$f: S \to S$$

$$f(z) = iz \qquad (z \in \mathbb{C})$$



Allowable functions $\mathcal{G}$

$\underset{\subseteq S}{A_1} \cong \underset{\subseteq S}{A_2}$ if there exists $f \in \mathcal{G}$

such that $\underbrace{f(A_1)}_{\{f(a) \,:\, a \in A_1\} \subseteq S} = A_2$

Rules for congruence:

1) $A \cong A$ for all $A \subseteq S$

2) If $A \cong B$ then $B \cong A$

3) $A \cong B,\ B \cong C \Rightarrow A \cong C$

$\mathcal{G}$ $\leftarrow$ These inspire restrictions on $\mathcal{G}$
(properties of)

1) $id: S \to S$ always is an element of $\mathcal{G}$.

$id(s) = s$

$id(A) = A$

2) If $f \in \mathcal{G}$ then f is invertible and $f^{-1} \in \mathcal{G}$.

$$\cancel{f:\mathbb{C}\to\mathbb{C} \qquad f(z)=0}$$

If $A \cong B$ there exists $f \in \mathcal{G}$ with $f(A)=B$.

Then
$$f^{-1}(f(A)) = f^{-1}(B)$$
$$A = \underbrace{f^{-1}}_{\in\, \mathcal{G}}(B)$$

3) If $f, g \in \mathcal{G}$ then $g \circ f \in \mathcal{G}$ also.

$$f(A) = B \qquad\qquad g(B) = C$$

$$g(f(A)) = g(B) = C$$
$$(g\circ f)(A) = C$$

Def: Let S be a nonempty set.

A family $\mathcal{G}$ of functions from S to S is called a <u>transformation group</u> if

1) $id \in \mathcal{G}$
2) Each $f \in \mathcal{G}$ is invertible and $f^{-1} \in \mathcal{G}$.
3) $\mathcal{G}$ is closed under composition

Def: A <u>geometry</u> is a pair $(S, \mathcal{G})$ where S is a set and $\mathcal{G}$ is a transformation group on S.

$f_0 \quad f_1 \quad f_2 \quad f_3$

$f_0 = id \qquad f_1(z) = iz \qquad f_2(z) = -z \qquad f_3(z) = -iz$

$f_k(z) = i^k z$

$S = \mathbb{C} \qquad \mathcal{G} = \{ f_0, f_1, f_2, f_3 \}$

Trivial group $\quad S \quad \mathcal{G} = \{ id \}$

For the trivial geometry a set is congruent only to itself.

Erlanger Program (Felix Klein)

2) Oriented Euclidean Geometry

$$S = \mathbb{C} \qquad \mathcal{G} = \{ f : f(z) = \underbrace{e^{i\theta} z + b}_{f_{\theta,b}}, \ \theta \in \mathbb{R}, \ b \in \mathbb{C} \}$$

$$\mathrm{id} = f_{0,0} \qquad e^{i0} z + 0 = 1 \cdot z = z$$

$$(f_{\psi,c} \circ f_{\theta,b})(z) = f_{\psi,c}\left(e^{i\theta} z + b\right)$$

$$= e^{i\psi}\left(e^{i\theta} z + b\right) + c$$

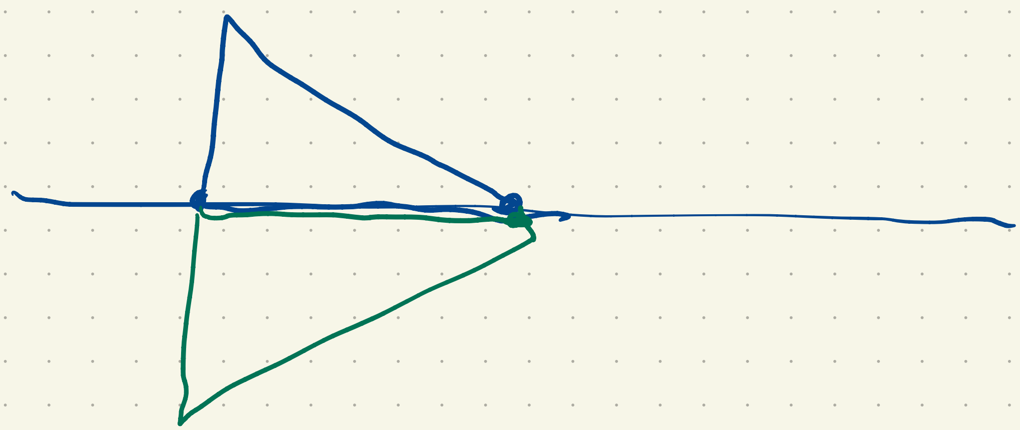
$$= e^{i\psi} e^{i\theta} z + e^{i\psi} b + c$$

$$= e^{\underbrace{i(\psi+\theta)}} z + \underbrace{e^{i\psi} b + c}$$

$$f_{\varphi+\theta,\, e^{i\varphi}b+c} \in \mathcal{G}$$

3) Euclidean Geometry

$$S = \mathbb{C} \qquad \mathcal{G} = \{ f_{\theta,b},\ \overline{f_{\theta,b}} : \theta \in \mathbb{R},\ b \in \mathbb{C} \}$$



4) Translational Geometry

$$S = \mathbb{C}$$

$$\mathcal{G} = \{ t_b : b \in \mathbb{C} \} \qquad t_b(z) = z + b$$

$$\mathcal{G}_{\text{trivial}} \subseteq \mathcal{G}_{\text{transl}} \subseteq \mathcal{G}_{\text{OEuc}} \subseteq \mathcal{G}_{\text{Euc}}$$

Notion of congruence gets easier $\longrightarrow$