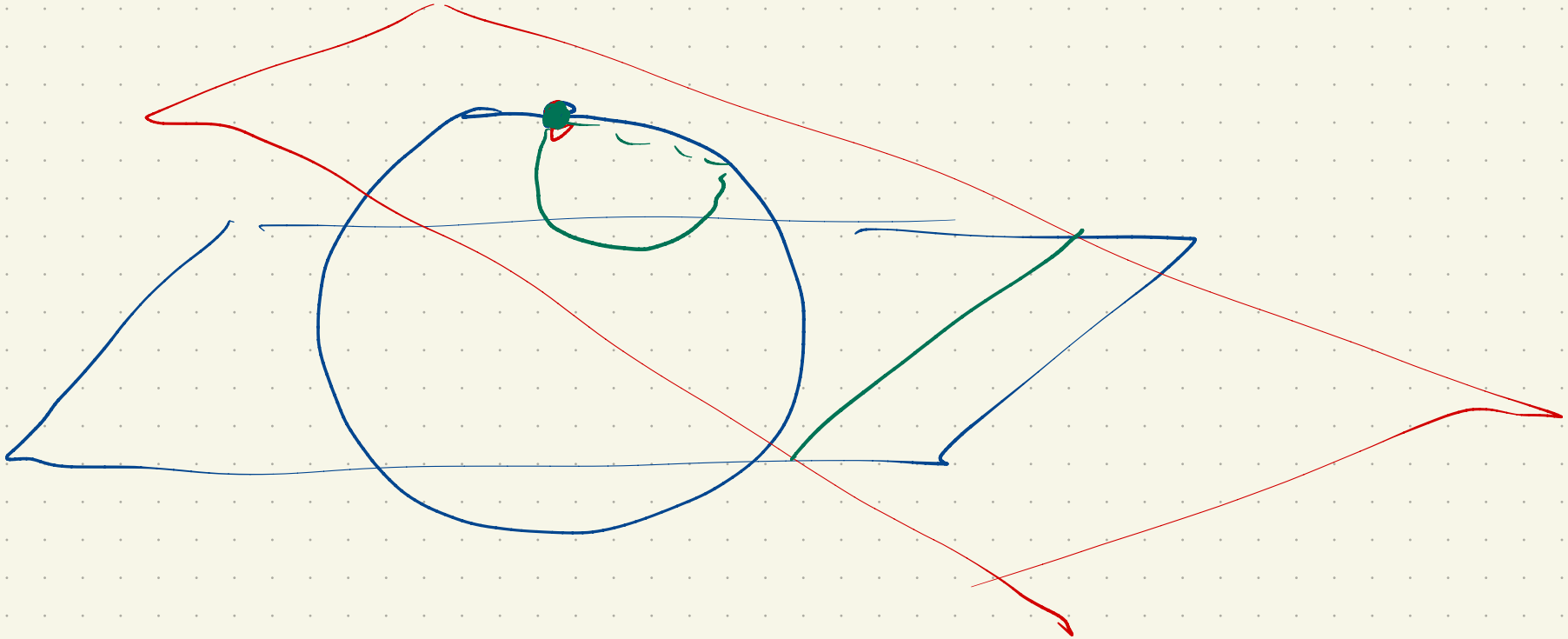
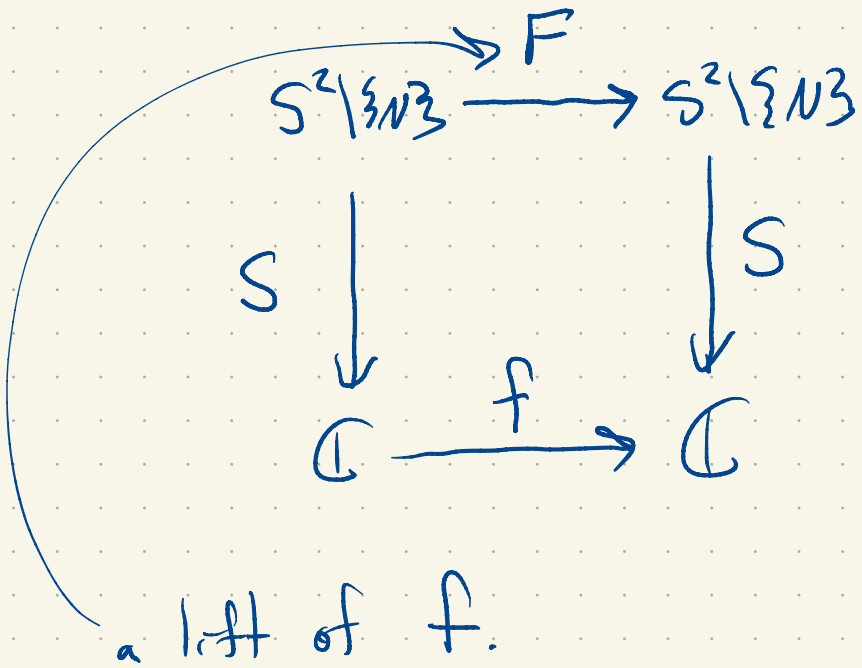
Stereographic projection is conformal. (!!)



$$S^2 \setminus \{N\} \xrightarrow{S} \mathbb{C}$$

$$\xleftarrow{S^{-1}} \text{Your job (HW)}$$



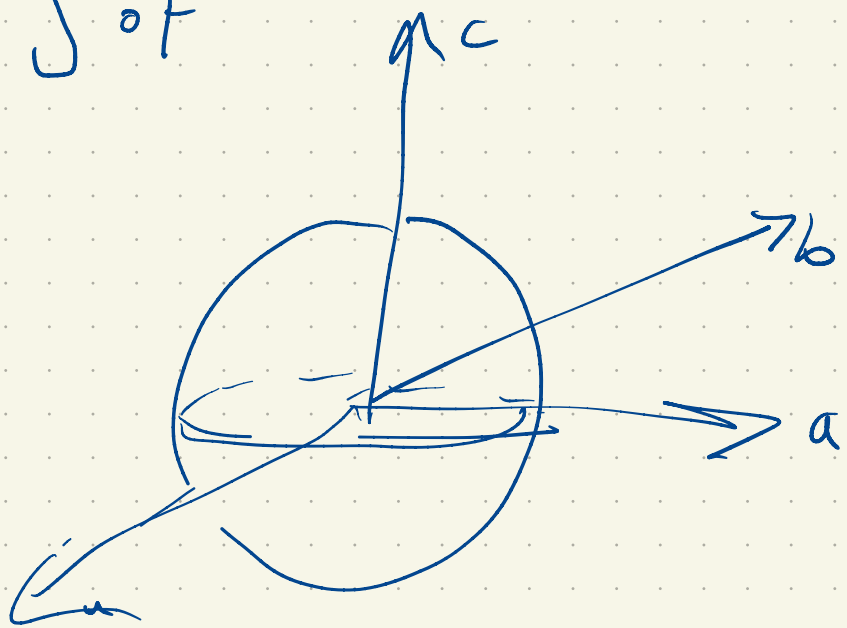
a lift of f.

$f(z) = z^2$

$f(z) = \bar{z}$

$$f \circ S = S \circ F$$

$f(z) = \bar{z}$



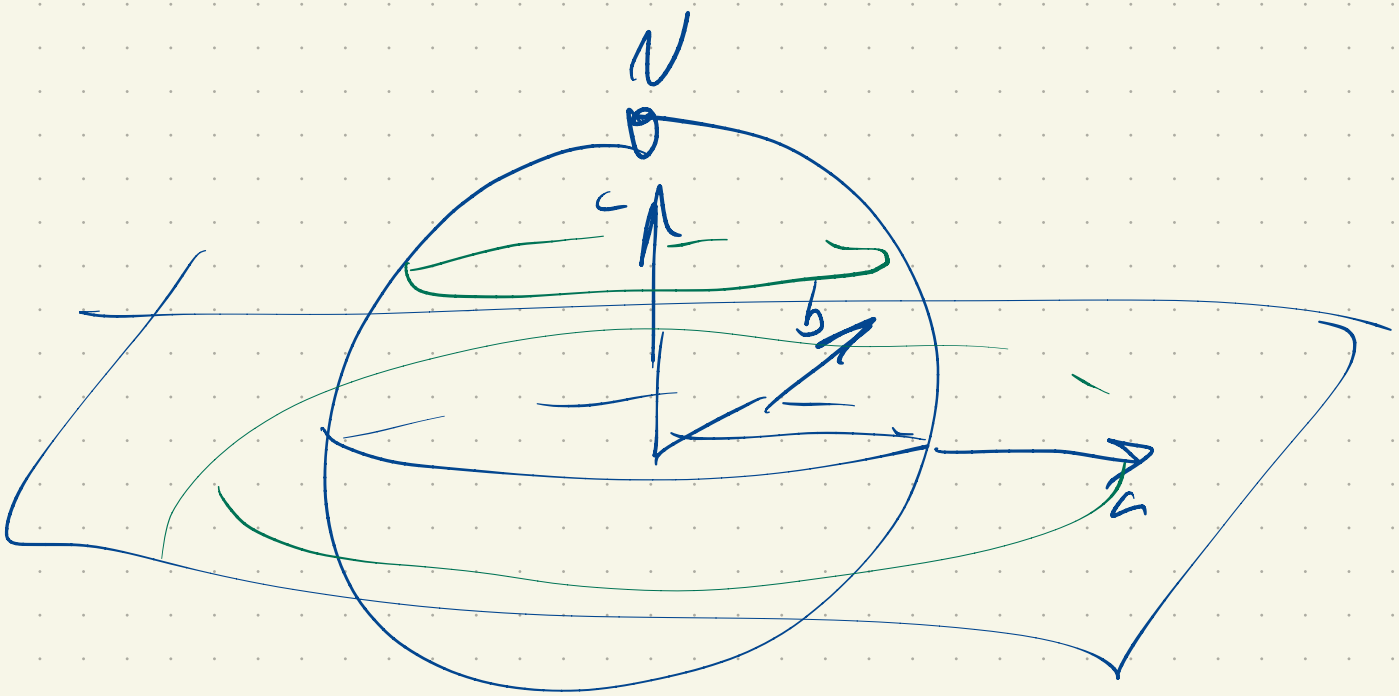
$$F(a,b,c) = (a, -b, c)$$

$$f \circ S = S \circ F$$

$$f \circ S(a,b,c) = f\left(\frac{a}{1-c} + \frac{ib}{1-c}\right) = \frac{a}{1-c} - \frac{ib}{1-c}$$

$$S \circ F(a,b,c) = S(a,-b,c) = \frac{a}{1-c} - \frac{ib}{1-c}$$

$$f(z) = e^{i\theta} z \qquad \theta \in \mathbb{R}$$



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta\, a - \sin\theta\, b \\ \sin\theta\, a + \cos\theta\, b \\ c \end{bmatrix}$$

↓

rotation matrix

about c-axis by angle θ

$$F(a,b,c) = (\cos\theta\, a - \sin\theta\, b,\ \sin\theta\, a + \cos\theta\, b,\ c)$$

$$S \circ F(a,b,c) = \frac{\cos\theta\, a - \sin\theta\, b}{1-c} + i\left(\frac{\sin\theta\, a + \cos\theta\, b}{1-c}\right)$$

$$f \circ S(a,b,c) = f\left(\frac{a}{1-c} + i\frac{b}{1-c}\right)$$

$$= e^{i\theta}\left(\frac{a}{1-c} + i\frac{b}{1-c}\right)$$

$$= (\cos\theta + i\sin\theta)\left(\frac{a}{1-c} + \frac{ib}{1-c}\right) = \checkmark$$

$$I(z) = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$$

$$f \circ S = S \circ F$$

$$|z|^2 = \frac{a^2+b^2}{(1-c)^2}$$

$$I(S(a,b,c)) = I\left(\frac{a}{1-c} + i\frac{b}{1-c}\right)$$

(z = the argument above)

$$= \frac{(1-c)^2}{a^2+b^2}\,\frac{1}{1-c}(a+ib)$$

$a^2+b^2+c^2=1$

$$= \frac{(1-c)^2}{1-c^2}\,\frac{1}{1-c}\,(a+ib)$$

$$= \frac{1-c}{1-c^2}\,(a+ib)$$

$$= \frac{1}{1+c}\,(a+ib)$$

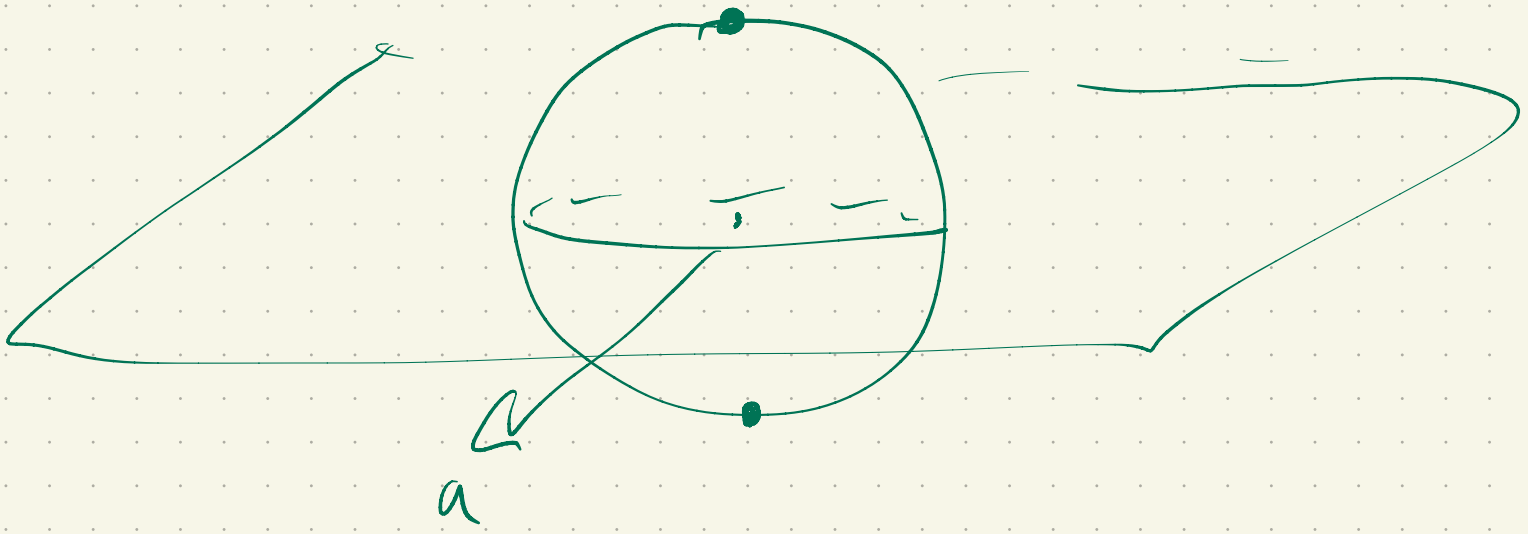
$$= \frac{a}{1+c} + i\,\frac{b}{1+c}$$

✓

$$F(a,b,c) = (a,b,-c)$$

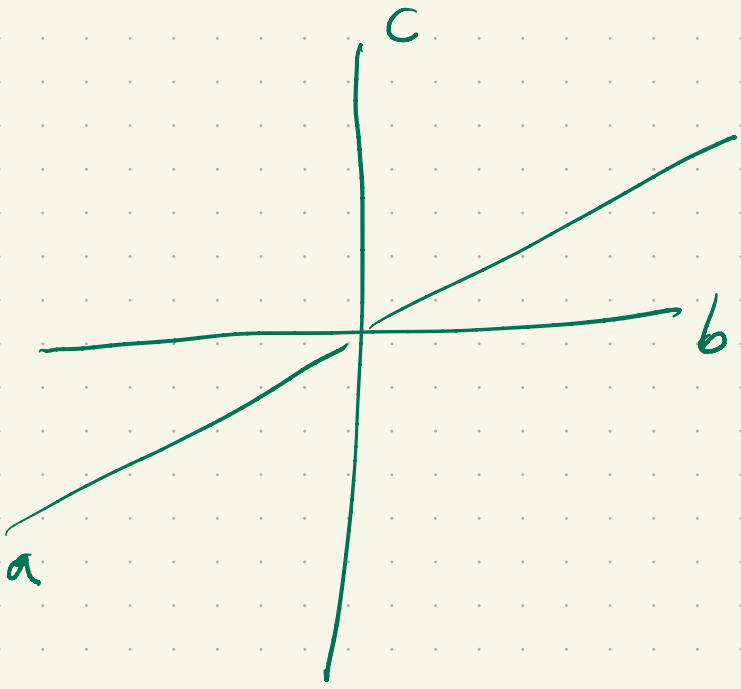
$$S \circ F(a,b,c) = S(a,b,-c) = \frac{a}{1-(-c)} + i\,\frac{b}{1-(-c)}$$

$$= \frac{a}{1+c} + i\,\frac{b}{1+c}$$



$$f(z) = \frac{1}{z} = \overline{I}(z)$$

$$F(a,b,c) = (a,-b,-c)$$



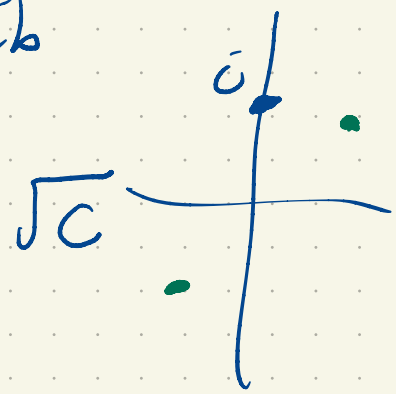
The lift of complex inversion is simply a rotation by π radians about the a axis.

$$\begin{array}{ccc} S^2 & \xrightarrow{F} & S^2 \\ s \downarrow & & \downarrow s \\ \mathbb{C} & \xrightarrow[f]{} & \mathbb{C} \end{array}$$

$$\begin{array}{ccc} S^2 \setminus \{N, S\} & & \\ s \downarrow & & \\ \mathbb{C} \setminus \{0\} & \longrightarrow & \mathbb{C} \setminus \{0\} \end{array}$$

$re^{i\theta}$

$z = a + ib$



$z^2 + c = 0$

$z^2 = -c$

$z = \pm\sqrt{-c}$