P1-P4

Complex Numbers

We add to $\mathbb{R}$ another number i $\quad i^2 = -1$

$$\underbrace{\text{Complex numbers}}_{\mathbb{C}} \qquad \underbrace{a + ib}_{z} \qquad a, b \in \mathbb{R}$$

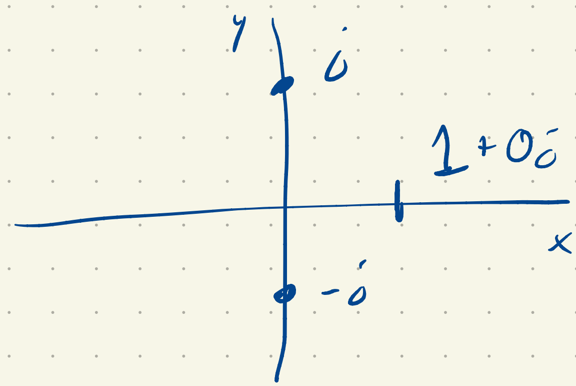
$$z = a + ib$$

$$\mathrm{Re}(z) = a$$

$$\mathrm{Im}(z) = b$$

We identify $\mathbb{R}$ with $\{a + 0i : a \in \mathbb{R}\}$

$$z = x + iy$$



$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\underline{(x_1 + iy_1) \cdot (x_2 + iy_2)} = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Exercise: $z_1 + z_2 = z_2 + z_1$ $\qquad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$$z_1 z_2 = z_2 z_1 \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$

$$kz = kx + iky \quad \text{if } k \in \mathbb{R}$$

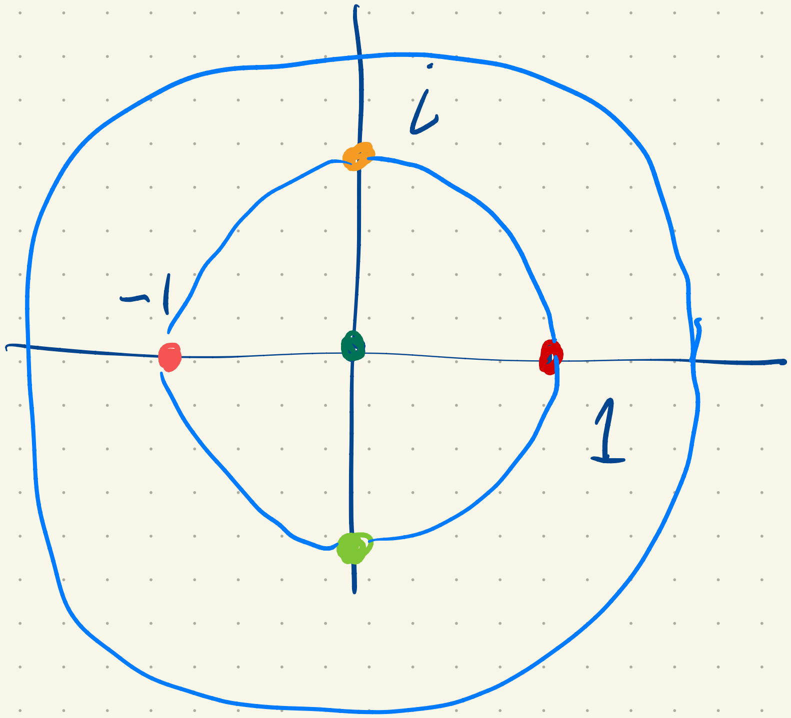
What is wz generally for $w, z \in \mathbb{C}$

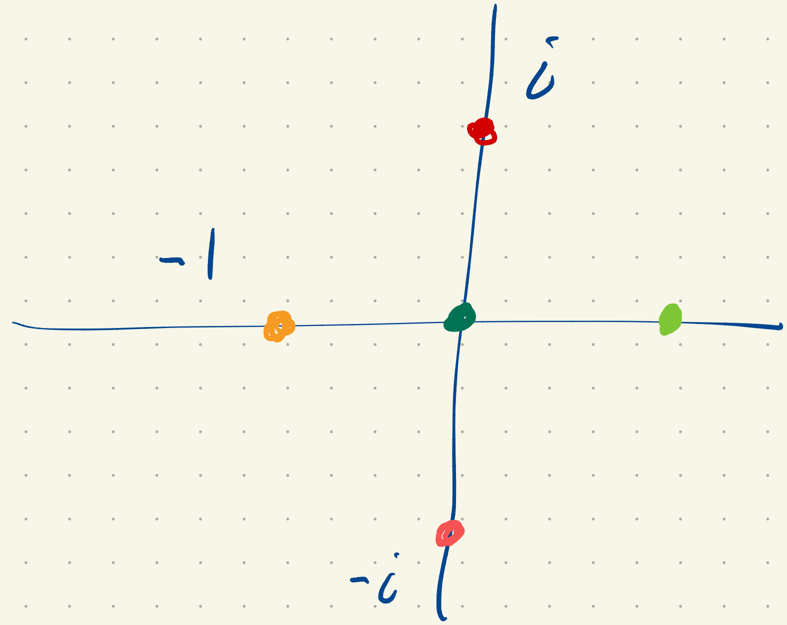
$$z = x + iy$$

$$iz = ix - y$$

$$= -y + ix$$

$z = i$





Looks like rotation by $\frac{\pi}{2}$ radians

$z = x + iy$ $\qquad |z| = \sqrt{x^2 + y^2}$

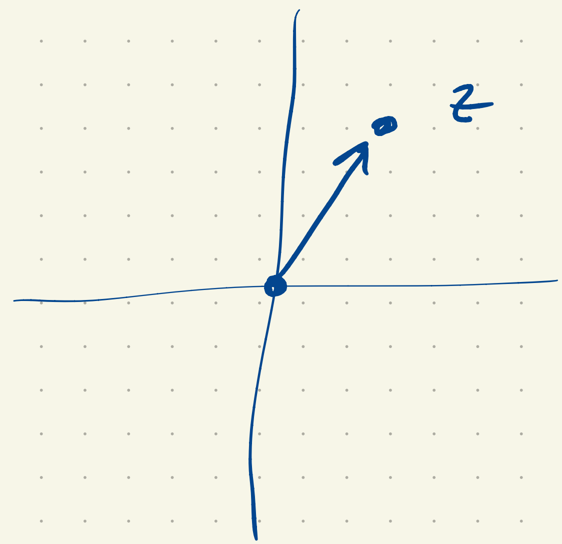
$|z|^2 = x^2 + y^2$

$|iz| = |y - ix| = \sqrt{y^2 + (-x)^2} = \sqrt{x^2 + y^2} = |z|$

$$|z_1 z_2| = |z_1|\,|z_2|$$

Verify on HW.

z, w dist between the two:



$$|w - z|$$

$$(z_1, z_2) \quad (w_1, w_2)$$

$$\sqrt{(w_1 - z_1)^2 + (w_2 - z_2)^2}$$

$$z = x + iy \qquad r = |z|$$

$$x^2 + y^2 = r^2$$

$$\underbrace{\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2}_{} = 1$$

There is an angle θ

with

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = r\cos\theta + i r\sin\theta = r\overbrace{\left[\cos\theta + i\sin\theta\right]}^{e^{i\theta}}$$

$$z_1 = r_1 \left(\cos\theta_1 + i\sin\theta_1\right)$$

$$z_2 = r_2 \left(\cos\theta_2 + i\sin\theta_2\right)$$

$$z_1 z_2 = r_1 r_2 \left[\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1\right)\right]$$

$$= \underbrace{r_1 r_2}_{R}\left[\cos\left(\underbrace{\theta_1+\theta_2}_{\Theta}\right) + i\sin\left(\theta_1+\theta_2\right)\right]$$

Angles sum and length multiply in complex multi

$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = R e^{i\Theta}$$

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

$$R \downarrow r_1 r_2 \qquad \Theta = \theta_1 + \theta_2$$

$$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\pi/2\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(really?)}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots$$

$$= \left(1 - \underline{\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

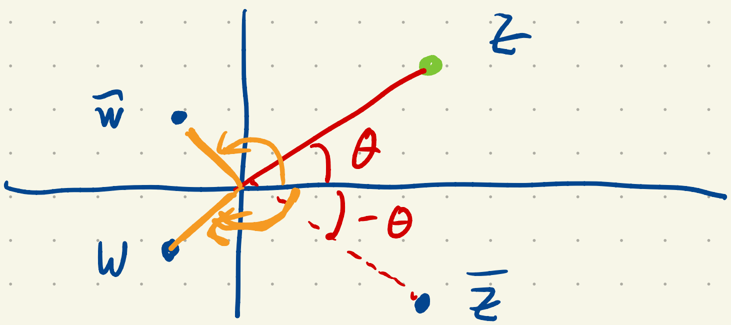
$$= \cos\theta + i\sin\theta$$

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = e^{i\theta_1 + i\theta_2}$$

$$e^x e^w = e^{x+w} \qquad z = a + ib \qquad e^z = e^a e^{ib}$$

$$e^{a+ib} = e^a e^{ib}$$

$z = x + iy \qquad \bar{z} = x - iy$ complex conjugate



a) $|\bar{z}| = |z|$

b) $|z|^2 = z\bar{z} = x^2 + y^2$

c) $\bar{\bar{z}} = z$

d) $\overline{zw} = \bar{z}\,\bar{w}$ verify

e) $\overline{re^{i\theta}} = \overline{r\cos\theta + i r\sin\theta}$

$= r\cos\theta - i r\sin\theta$

$= r\cos(-\theta) + i r\sin(-\theta)$

$= re^{-i\theta}$

Geometric operations via complex numbers:

$$f : \mathbb{C} \to \mathbb{C}$$

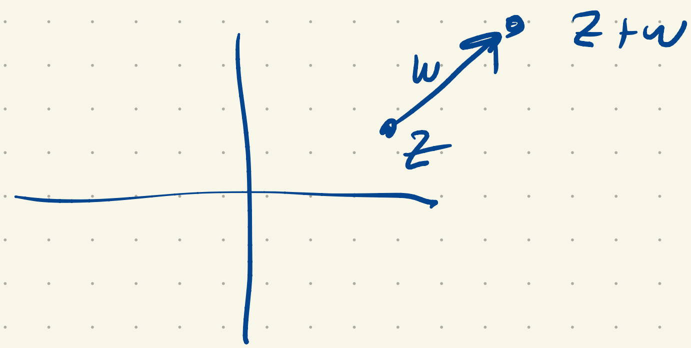
a) Rotate $\mathbb{C}$ about 0 by $\cancel{\frac{\pi}{2}}$ θ radians

$$f(z) = i z$$

$$e^{i\pi/2}$$

$$f(z) = e^{i\theta} z$$

b) translation by $w \in \mathbb{C}$



$$f(z) = z + w$$

c) reflection about x-axis

$$f(z) = \overline{z}$$

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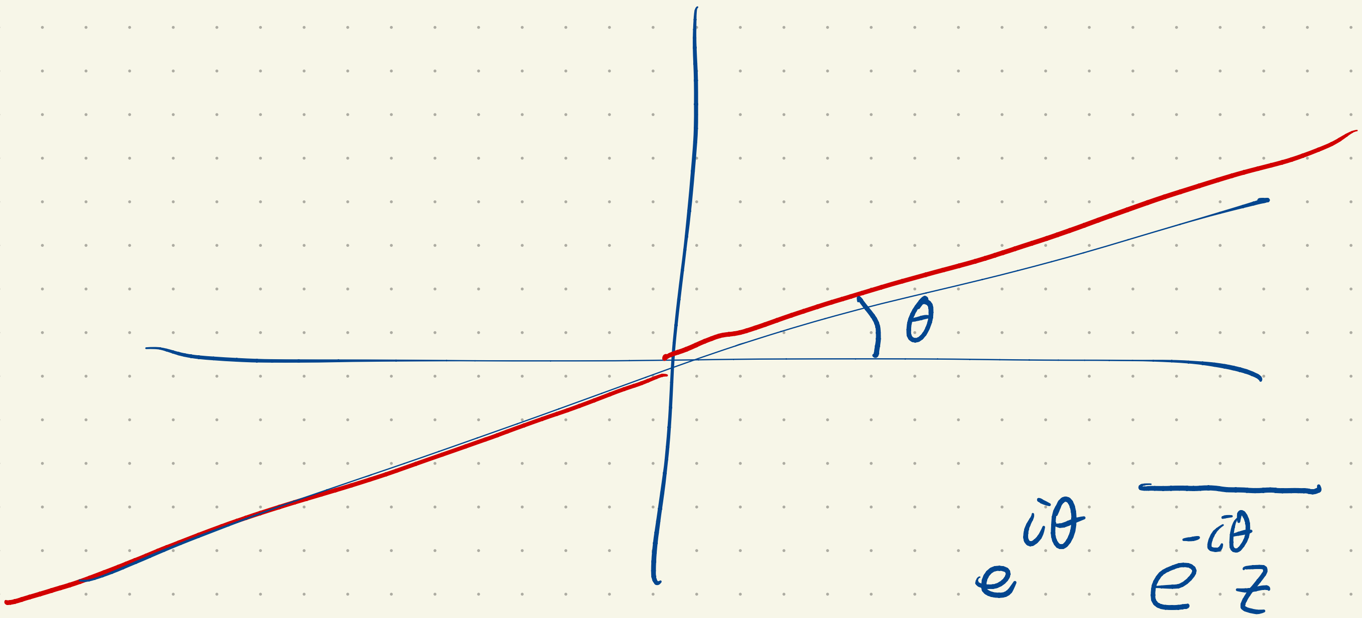
rotate by θ radians around w

$$e^{i\theta}(z-w) + w$$

$$e^{i\theta}z + w(1-e^{i\theta})$$



$$e^{\pi i} = -1$$

$$e^{i\theta}\,\overline{e^{-i\theta} z} = e^{i\theta} e^{i\theta}\bar{z}$$

$$= e^{2i\theta}\bar{z}$$

$$-\bar{z}$$

$$-(x - iy) = -x + iy$$