The integers modulo n, $\mathbb{Z}_n$, consist of the elements $0, 1, \ldots, n-1$. To add or multiply two numbers a and b, you add or multiply them as usual for integers, and then you find the remainder of your answer upon division by n. If n is prime, you can show that $\mathbb{Z}_n$ is an algebraic field: it has division as well as multiplication.

The two smallest cases are $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Here are the addition and multiplication tables:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

+	0	1	2
0	0	1	2
1	1	2	3
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

1. Draw a picture of $\mathbb{Z}_2$ and a picture of $(\mathbb{Z}_2)^2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Yeah, it's just dots. But try to arrange them in a way insipred by how you draw, e.g., $\mathbb{R}^2$.

2. Recall that slope is rise over run. If your rise comes from $\mathbb{Z}_2$ and your run comes from $\mathbb{Z}_2$, how many slopes can you make? We'll stay far, far away from $0/0$ of course; that's not a slope.

3. To make a line in $\mathbb{Z}_2$ you start at your favorite point and then add 1 to x and adjust the y coordinate by the slope m. Now repeat to get the whole line. Well, that's true except for 'vertical' lines. To make these add one to y and don't adjust x. Anyway, starting at the point $(0, 0)$, how many different lines are there?

4. How many lines are there in $(\mathbb{Z}_2)^2$? Be careful you don't miss one!

5. Given a line in $(\mathbb{Z}_2)^2$, how many lines are parallel to it?

6. Parallel lines should meet at points at infinity. How many points at infinity should there be in the corresponding projective plane, $\mathbb{Z}_2\mathbb{P}^2$?

7. How many points are there in $\mathbb{Z}_2\mathbb{P}^2$?

8. Including the line at infinity, how many lines are there in $\mathbb{Z}_2\mathbb{P}^2$?

9. Try to draw a picture of $\mathbb{Z}_2\mathbb{P}^2$ with all of its points and lines.

10. An abstract projective plane satisfies three axioms:

1. Every pair of two points is on exactly one line.
2. Every pair of two lines intersects in exactly one point.
3. There exist four points, no three of which are on a common line.

Verify that $\mathbb{Z}_2\mathbb{P}^2$ is an abstract projective plane.

11. Draw a picture of $(\mathbb{Z}_3)^2 = \mathbb{Z}_3 \times \mathbb{Z}_3$.

12. A line in $(\mathbb{Z}_3)^2$ is a set of the form $L_{x,v} = \{x + sv : s\mathbb{Z}_3\}$ where $x, v \in (\mathbb{Z}_3)^2$ and where $v \neq 0$. Explain why if v is a vector (a, b) with $a \neq 0$ we can write $L_{x,v} = L_{x,\hat{v}}$ where $\hat{v} = (1, c)$ for some $c \in \mathbb{Z}_3$. Can you spot a crucial algebraic operation that you use in this argument?

13. How many different slopes can a line through the origin in $(\mathbb{Z}_3)^2$ have?

14. How many lines through the origin are there in $\mathbb{Z}_3\mathbb{P}^2$?

15. How many points at ∞ should there be in $\mathbb{Z}_3\mathbb{P}^2$? How many points are there in $\mathbb{Z}_3\mathbb{P}^2$?

16. Given a line in $(\mathbb{Z}_3)^2$, how many lines seem to be parallel to it?

17. How many lines are there in $(\mathbb{Z}_3)^2$? How many lines are there in $\mathbb{Z}_3\mathbb{P}^2$?

18. Does $\mathbb{Z}_3\mathbb{P}^2$ satisfiy the axioms of a projective plane?

19. Show that in $(\mathbb{Z}_4)^2$ there is a pair of points with **two** distinct lines between them. No projective plane here...