1. Determine the values of θ for which the θ method is L-stable.

2. Consider the function $f(x) = x - x^2$ on the interval $[0,1]$.

a) Compute the Fourier sine series coefficients of $f(x)$.

b) The function $f(x)$ looks pretty darn smooth. In class we saw that smoothness should be reflected in the rate of decay of the Fourier coefficients. But the coefficients of $f(x)$ don't decay very fast, only $O(N^3)$. Why doesn't this contradict the theorems we saw in class?

c) It can be shown that $\sum_{k=N}^{\infty} \frac{1}{k^3} \leq \frac{1}{2N^2}$. Use this result to show that if $s_N(x)$ is the partial sum of the Fourier sin series of $f(x) = x - x^2$ with N terms, then

$$\max_{0\leq x\leq 1} |f(x) - s_N(x)| \leq \frac{4}{\pi^3 N^2}$$

d) Generate a convincing plot that shows that the error between $f(x)$ and the partial sum $s_N(x)$ of the Fourier sine series N terms converges $O(N^{-2)})$. You must show the code used to generate the plot.

3. Suppose u is a solution of $u_t = u_{xx}$ for $0 \leq x \leq 1$ and $t \geq 0$ with boundary condition $u|_{x=0,1} = 0$.

a) Suppose that u has j continuous time derivatives and $2j$ continuous space derivatives everywhere on its domain for some $j = 1, 2, 3, \ldots$. Show that $(\partial_x)^{2j} u = 0$ at $x = 0, 1$.

b) Suppose you solve this problem with initial data $u(x,0) = x - x^2$. Does the solution have one continuous time derivative and two continuous space derivatives everywhere on its domain? Justify you answer briefly.

4. Recall the partial sums $s_N(x)$ from problem 2. Suppose $u_N(x,t)$ is the solution of the heat equation $u_t = u_{xx}$ for $0 \leq x \leq 1$ with $u_N(x,0) = s_N(x)$ with Dirichlet boundary conditions. Suppose $u(x,t)$ the the solution of the heat equation with the same boundary conditions but with $u(x,0) = x - x^2$. Show that $|u_N(x,t) - u(x,t)| < 10^{-7}$ for all $x \in [0,1]$ and all $t \geq 0$ if $N = 1200$.

5. Suppose we wish to solve $u_t = u_{xx}$ with homogeneous Dirichlet boundary conditions and $u(x,0) = x - x^2$. The aim of this exercise (and indeed this entire assignment) is to show that if the solution of the heat equation isn't smooth, then the order of accuracy of your numerical solution can be reduced from the accuracy expected using arguments that use smoothness.

We don't know the exact solution of the heat equation with this initial condition. But by the previous problem, we know that we can compute an approximate solution with a known error by using the series solution with 1200 terms. So this will play the role of the "exact" solution, which is good enough until we see errors on the order of 10^{-7}.

We are going to compare solving the heat equation with homogeneous Dirichlet boundary conditions with initial condition $f_1(x) = x - x^2$ and with initial condition $f_2(x) = \sin(\pi x)/4$.

a) Generate a graph of $f_1(x)$ and $f_2(x)$ for $0 \le x \le 1$. This step is just to convince you that these initial conditions look "close" to each other.

b) For $N = 50, 100, 500, 1000, 5000$ and $M = 2N$, generate a solution of the heat equation using backwards Euler and initial condition $f_2(x) = \sin(\pi x)/4$. Then compute the error at the first time step (i.e. at the first time beyond $t = 0$). Generate a log-log plot of the error versus N and compute the order of convergence.

c) Repeat the above but measuring error at the final time step. Why do you see the two orders of convergence you observe in this part and in the previous part? Why are they different?

d) Repeat parts b) and c), but with Crank Nicolson.

e) Now generate the same log-log plots of error (with computed orders of convergence) when the initial condition is $f_1(x) = x - x^2$. There should be four log-log plots (first time step and last time step for each of backward Euler and Crank Nicolson).

f) Discuss the differences you see between the various convergence plots for f_1 and their corresponding plots for f_2.