1. Consider the heat equation $u_t = \kappa u_{xx}$ for $\kappa > 0$, $x \in [0,1]$, and Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Suppose we have initial condition $u(x,0) = sin(5\pi x)$.

a) Find an exact solution to this problem.

b) Implement the backward Euler (BE) method to solve this heat equation problem. Specifically, use diffusivity $\kappa = 1/20$ and final time $T = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, which is a linear system, but you should use sparse storage and an efficient linear solver (backslash in MATLAB will work).

c) Suppose the timestep k and the space step h are related by $k = 2h$. What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from a), at the final time, and the infinity norm $||\cdot||_\infty$, and $h = 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$. Make a log-log convergence plot of h versus the error.

d) Repeat parts b) and c) but with the trapezoidal rule instead of BE. (That is, implement and measure the convergence rate of Crank-Nicolson, with everything else the same.)

2. Consider the PDE

$$u_t = \partial_x(p(x)u_x)$$

where $p(x)$ is a given function. We wish to solve the PDE on the region $0 \le x \le 1$, $0 \le t \le T$ with $u = 0$ at $x = 0, 1$ We will apply the following finite difference scheme to it:

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}[(u_{i+1,j} - u_{i,j})p_{i+\frac{1}{2}} - (u_{i,j} - u_{i-1,j})p_{i-\frac{1}{2}}]$$

where $p_{i\pm\frac{1}{2}} = p(x_i \pm h/2)$.

a) Estimate the local truncation error in terms of powers of h and k and in terms of derivatives of u and derivatives of p. I'm looking for an answer akin to the estimate we derived for the heat equation of the form

$$|\tau| \le \max|u_{xxxx}|\left[\frac{k}{2} + \frac{h^2}{h}\right]$$

that we derived for the heat equation with no forcing term.

b) Show that the method is convergent, assuming $0 < p(x)k < h^2/2$. You will want to revist the proof from class that the explict method for the standard heat equation is convergent.

3.

a) Let

$$A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Compute $||A||_1$ and $||A||_\infty$.

b) Estimate $||A||_2$ as follows. Computer generate a figure containng the boundary of $A(B_1)$, where B_1 is the Euclidean ball of radius 1. Then use the figure to estimate the norm.

c) Suppose A is an $n \times n$ matrix, and choose $p \in [1, \infty]$. Show that $||A||_p = 0$ if and only if A is the 0 matrix.

d) For vectors in $\mathbb{R}^n$, it is known that $||x + y||_p \leq ||x||_p + ||y||_p$ for any $p \in [1, \infty]$. This is the triangle inequality, and you need not prove it. But using this fact, show that the triangle inequality also holds for matrix norms $|| \cdot ||_p$ for p in the same range.

4. Text, problem 3.7