1. Consider the matrix

$$A = \begin{pmatrix} 23 & -8 & 4 \\ 21 & -8 & 5 \\ -126 & 42 & -19 \end{pmatrix}.$$

a) Show that $v_1 = [-1,-2,3]$, $v_2 = [1,3,0]$ and $v_3 = [0,1,2]$ are eigenvectors of A, and determine their associated eigenvalues.

b) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

c) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_2 + v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

d) Determine the exact solution of

$$u' = Au$$

with initial condition $u(0) = [1,5,5]$.

2. Suppose you wish to apply the RK4 method to solve the ODE of the previous problem. What is the largest time step you can use before issues concerning absolute stability arise in your solution?

3.

a) Use your Newton solver from last week's homework to implement the trapezoidal rule for solving systems of ODEs.

b) Determine the exact solution to the problem

$$\begin{aligned} u' &= 1 \\ v' &= v - u^2 \end{aligned} \tag{1}$$

with initial condition $u(0) = 0$ and $v(0) = 1$.

c) Test your solver against the previous exact solution and confirm that it has the predicted order of accuracy.

4. Implement the explict method for solving the heat equation with right-hand side function

$$u_t = u_{xx} + f$$

on $0 \le x \le 1$ and $0 \le t \le T$. You function should have the following signature:

forcedheat(f,u0,N,M)

where

- $f(x,t)$ is a function and provedes the desired forcing term
- $u0(x)$ is a function and provides the desired initial condition.
- $N+1$ is the number of interior spatial steps
- M is the number of time steps

It should return (x,t,u) where x is an array of grid coordinates that includes 0 and 1, t is a vector of t coordinates that includes 0 and T, and where u is an $(N+2)\times(M+1)$ matrix where column j encodes the solution at time t_j.

Test your code as follows

- Compute what f is if the solution is $u(t,x) = \sin(t)x(1-x)$.
- Now, working on $0 \le x \le 1$ and $0 \le t \le 2\pi$ compute solutions with this forcing term and compare your solution with the exact solution. By working with various grid sizes, confirm that your code has the expected order of convergence.