1.

a) Make a graph of the boundary of the absolute stability region for the Runge-Kutta RK4 method on page 24.

b) Apply the RK4 method to $u' = 30u(1-u)$ with u(0) = 0.1 on the interval $0 \leq t \leq 2$. Use 14, 17 and 25 steps. For each run graph the numerical solution and the exact solution on the same plot.

c) Explain the previous sequence of graphs in terms of the ODE and the plot from part a. Your answer should contain a quantitative explanation for why the transiton occurs at the value of h you observe.

2. Newton's method can be used to solve

$$f(x) = 0$$

where $x \in \mathbb{R}^n$ and $f(x) \in \mathbb{R}^n$. Starting from an initial guess x_k,

$$x_{k+1} = x_k - Df(x_k)^{-1} f(x_k).$$

Here, $Df(x)$ is the Jacobian matrix

$$Df_{ij} = \frac{\partial f_i}{\partial x_j}$$

Implement Newton's method for systems. Your function should take as arguments f, Df and x_0 (an initial guess). It should terminate whenever either

- $|f(x)|_\infty$ is less than a specified tolerance
- $|f(x)|_\infty$ is less than a sepcified fraction of $|f(x_0)|_\infty$.

These tolerances should be specified with optional arguments as used in your language of choice, with values of 10^{-8} as the default.

Your function should return the estimated root and, as a diagnostic, the number of iterations needed to find the root.

Test your code against solving the simultaneous equations $x^2 + y^2 = 1$ and $x = y$ starting from $x = 0$, $y = 3$. Report the root found and the number of iterations needed to find it.

3. The energy for the heat equation $u_t = u_{xx}$ for $0 \leq x \leq 1$ is

$$E(t) = \frac{1}{2} \int_0^1 (u_x(x,t))^2 \, dx.$$

a) Assuming that at $x = 0$ and at $x = 1$ u satisfies either a homogeneous Dirichlet condition or a homogeneous Neumann condition, show that

$$\frac{d}{dt}E(t) \le 0.$$

Hint: Take a time derivative, use the PDE, and integrate by parts.

b) Conclude that the only solution of $u_t = u_{xx}$ with $u = 0$ at $t = 0$, and at $x = 0$ and $x = 1$ is the zero solution.

4. The backwards heat equation reads

$$u_t = -u_{xx},$$

so all that differs is a sign on the right-hand side. But this sign makes all the difference.

We will work with this equation for $0 \le x \le 1$ and $0 \le t \le 1$, and with homogeneous Dirichlet boundary conditions, so $u = 0$ at $x = 0$ and $x = 1$.

a) Show that

$$v(t) = \sin(k\pi x)e^{k^2\pi^2 t}$$

is a solution of the PDE and the boundary conditions.

b) For each $\epsilon > 0$, find a solution of the PDE and boundary conditions that satisfies $|u(0,x)| < \epsilon$ at each x, but $|u(1,x)| \ge 1$ at some x.

c) Suppose you wish to find the solution u of the backwards heat equation with initial condition u_0. But you don't know u_0 exactly, you know $\hat{u}_0$, and that $|u_0(x) - \hat{u}_0(x)| < 10^{-47}$ at every x. So you solve the backwards heat equation for $\hat{u}$ instead. Find an L such that $|u(x,1) - \hat{u}(x,1)| < L$ for all x, or explain why no such L exists.

5. [Due next time, but start now] Implement the explict method for solving the heat equation with right-hand side function

$$u_t = u_{xx} + f$$

on $0 \le x \le 1$ and $0 \le t \le T$. You function should have the following signature:

forcedheat(f,u0,N,M)

where

- $f(x,t)$ is a function and provedes the desired forcing term
- $u0(x)$ is a function and provides the desired initial condition.
- $N + 1$ is the number of interior spatial steps
- M is the number of time steps

It should return (x, t, u) where x is an array of grid coordinates that includes 0 and 1, t is a vector of t coordinates that includes 0 and T, and where u is an $(N+2) \times (M+1)$ matrix where column j encodes the solution at time t_j.

Test your code as follows

- Compute what f is if the solution is $u(t,x) = \sin(t)x(1-x)$.
- Now, working on $0 \le x \le 1$ and $0 \le t \le 2\pi$ compute solutions with this forcing term and compare your solution with the exact solution. By working with various grid sizes, confirm that your code has the expected order of convergence.