In the first part of this worksheet we will get to know a method for computing an approximation of $\sqrt{2}$ to many digits of accuracy using only addition, subtraction, multiplication and division, and indeed using only a few such operations.

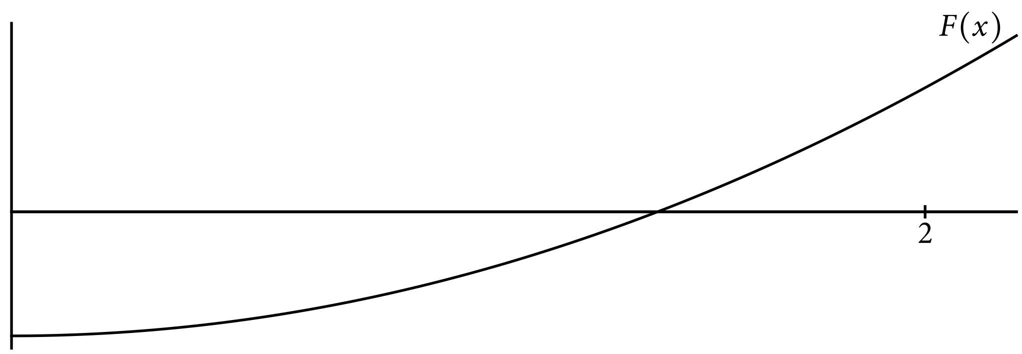
1. Consider the function

$$F(x) = x^2 - 2.$$

If we solve $F(a) = 0$ for some $a \geq 0$, what is the value of a?

2. Find the linearization $L(x)$of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

3. I've graphed $F(x)$ for you below. Add to this diagram the graph of $L(x)$.



4. Find the number x_1 such that $L(x_1) = 0$.

5. What good is the number x_1? Keep in mind that you want to solve $F(x) = 0$. You solved $L(x) = 0$ instead.

6. In the diagram above, label the point x_1 on the x-axis.

7. Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$.

8. Add the graph of this new linearization to your diagram on the first page.

9. Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

10. To how many digits does x_2 agree with $\sqrt{2}$

11. Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$.
- Compute $F'(x_k)$.
- Compute the linearization of $F(x)$ at $x = x_k$.

  $L(x) =$

- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can.

12. Starting with $x_0 = 2$, compute x_1 and x_2 with your shiny new formula. Verify that they agree with your earlier expressions for x_1 and x_2.

13. Compute x_4. To how many digits does it agree with $\sqrt{2}$?

Newton's Method In General

We wish to solve $F(x) = 0$ for a differentiable function $F(x)$. We have an initial estimate x_0 for the solution.

14. Try to solve

$$e^{-x} - x = 0$$

by hand.

15. Explain why there is a solution between $x = 0$ and $x = 1$.

16. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 10 decimal places of accuracy.