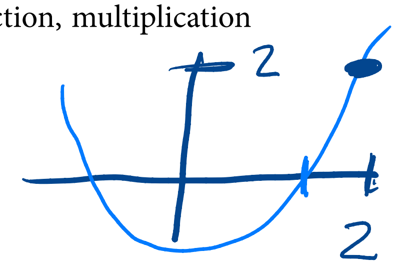
In the first part of this worksheet we will get to know a method for computing an approximation of $\sqrt{2}$ to many digits of accuracy using only addition, subtraction, multiplication and division, and indeed using only a few such operations.



1. Consider the function

$$F(x) = x^2 - 2.$$

If we solve $F(a) = 0$ for some $a \geq 0$, what is the value of a?

$a = \sqrt{2}$

2. Find the linearization $L(x)$ of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

$F'(2) \to$ slope of tangent line

$(2, F(2))$

$y - 2 = 4(x-2)$

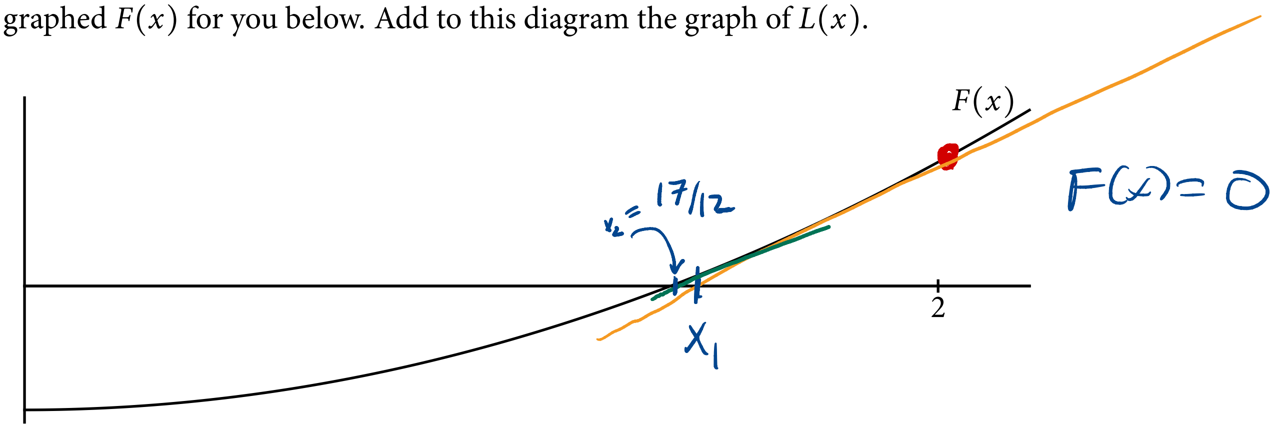
$y = 2 + 4(x-2) = L(x)$

$F'(x) = 2x$

$F'(2) = 4$

$(2, F(2)) = (2,2)$

3. I've graphed $F(x)$ for you below. Add to this diagram the graph of $L(x)$.



$F(x) = 0$

4. Find the number x_1 such that $L(x_1) = 0$.

$L(x_1) = 0$

$2 + 4(x_1 - 2) = 0$

$4(x_1 - 2) = -2$

$x_1 - 2 = -\frac{1}{2}$

$x_1 = 2 - \frac{1}{2} = \frac{3}{2} = 1.5$

5. What good is the number x_1? Keep in mind that you want to solve $F(x) = 0$. You solved $L(x) = 0$ instead.

Since $F(x) \approx L(x)$ for x near 2, if $L(3/2) = 0$, $F(3/2) \approx 0$.

6. In the diagram above, label the point x_1 on the x-axis.

$y_0 + m(x - x_0)$

7. Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$. $= \frac{3}{2}$

$F(x_1) = x_1^2 - 2 = \left(\frac{9}{4}\right) - 2 = \frac{1}{4}$

$F'(x_1) = 2x_1 = 2 \cdot \frac{3}{2} = 3 \qquad L(x) = \frac{1}{4} + 3\left(x - \frac{3}{2}\right)$

8. Add the graph of this new linearization to your diagram on the first page.

9. Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

$\frac{1}{4} + 3\left(x_2 - \frac{3}{2}\right) = 0$

$3\left(x_2 - \frac{3}{2}\right) = -\frac{1}{4}$

$x_2 = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}$

10. To how many digits does x_2 agree with $\sqrt{2}$

$\frac{17}{12} = \underbrace{1.41}6\ldots \qquad \sqrt{2} = \underbrace{1.41}4\ldots$

11. Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$. $x_k^2 - 2$
- Compute $F'(x_k)$. $2x_k$
- Compute the linearization of $F(x)$ at $x = x_k$.

$L(x) = (x_k^2 - 2) + 2x_k(x - x_k)$

- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can.

$L(x_{k+1}) = 0$

$(x_k^2 - 2) + 2x_k(x_{k+1} - x_k) = 0$

$$2x_k\left(x_{k+1}-x_k\right) = -\left(x_k^2-2\right)$$

$$x_{k+1} = x_k - \frac{\left(x_k^2-2\right)}{2x_k}$$

$$= x_k - \frac{x_k}{2} + \frac{1}{x_k}$$

$$x_{k+1} = \underbrace{\frac{x_k}{2} + \frac{1}{x_k}}_{\Phi(x_k)}$$

$$x_0 = 2$$

$$x_1 = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{3/2}{2} + \frac{1}{3/2} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$$

12. Starting with $x_0 = 2$, compute x_1 and x_2 with your shiny new formula. Verify that they agree with your earlier expressions for x_1 and x_2.

See above

13. Compute x_4. To how many digits does it agree with $\sqrt{2}$?

$x_1 = \frac{3}{2}$, $x_2 = \frac{17}{12}$

$x_3 = 1.4142156862774...$

$x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k}$

$x_4 = 1.41421356237468 99$

$\sqrt{2} = 1.41421356 2|$

Newton's Method In General

We wish to solve $F(x) = 0$ for a differentiable function $F(x)$. We have an initial estimate x_0 for the solution.

Linearization $L(x) = F(x_0) + F'(x_0)(x - x_0)$

$L(x) = 0$

$F(x_0) + F'(x_0)(x - x_0) = 0$

$F'(x_0)(x - x_0) = -F(x_0)$

$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$

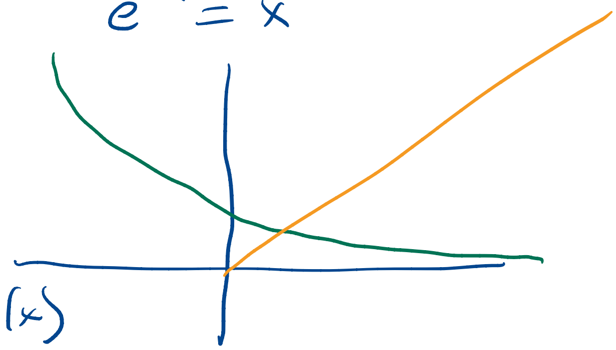
$x_{k+1} = x_k - \underbrace{\frac{F(x_k)}{F'(x_k)}}_{\Phi(x_k)}$

14. Try to solve

$$e^{-x} - x = 0$$

by hand.

$e^{-x} = x$



$e^{-x} = x$

$\ln(e^{-x}) = \ln(x)$

$-x = \ln(x) \rightarrow x = -\ln(x)$

15. Explain why there is a solution between $x = 0$ and $x = 1$.

$F(x) = e^{-x} - x$

$F(0) = 1 - 0 = 1$

$F(1) = e^{-1} - 1 < 0$

Intermediate Value Theorem!

16. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 10 decimal places of accuracy.

$F(x) = e^{-x} - x$ want $F(x) = 0$

$$x_{k+1} = \boxed{x_k - \frac{F(x_k)}{F'(x_k)}} \rightarrow \Phi(x_k)$$

$F'(x) = -e^{-x} - 1$

$$\Phi(x) = x - \frac{e^{-x} - x}{-e^{-x} - 1} = x + \frac{e^{-x} - x}{e^{-x} + 1}$$

$$= \frac{x+1}{e^{x}+1}$$

$x_0 = 1$

$$x_1 = \Phi(x_0) = \frac{2}{e^1 + 1} = \frac{2}{e+1} = 0.537883$$

$x_0 = 1$

$x_1 = 0.5378828427\ldots$

$x_2 = 0.566986991 4\ldots$

$x_3 = 0.567143285 9\ldots$

$x_4 = 0.\underbrace{56714329}04$

these look trusted