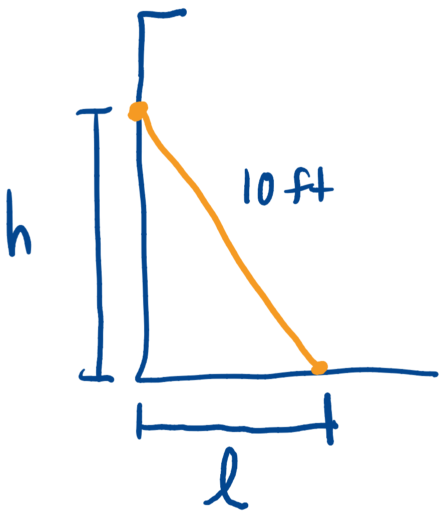
There is a class of problems in calculus, known as related rate problems. Here's the idea. You know the rate of change (often with respect to time) of one quantity, such as the volume of a spherical balloon. You want to know the rate of change of some other related quantity (e.g. the radius of the balloon). Here are the steps you take to solve a problem like this:

1. Identify the quantity you already know a rate of change of (say, V, so you know dV/dt).
2. Identify the quantity you want a rate of change of (say, r, so you want dr/dt).
3. Find an equation that relates the two quantities (V and r). This can be the hard part. Drawing a picture can help.
4. Now take a derivative with respect to t of both sides of the equation, treating both V and r as functions of t.
5. Substitute all known data into the result (typically V, r and dV/dt) to determine dr/dt.

We'll repeat this procedure with a bunch of examples.

1. A 10-foot ladder is sliding down a wall. If the bottom of the ladder slides along the floor at a rate of 1 ft/s, how fast does the top of the ladder slide down the wall when the bottom of the ladder is 6 feet from the wall?



t: time, seconds

l: dist from wall to foot of ladder, ft

h: dist from floor to top of ladder, ft

1) we know $\frac{dl}{dt}$

2) we want $\frac{dh}{dt}$

3) need a relationship between l and h

Pythagorus: $l^2 + h^2 = 10^2$

4) $\frac{d}{dt}(l^2 + h^2) = \frac{d}{dt} 100$

$$2l\frac{dl}{dt} + 2h\frac{dh}{dt} = 0$$

5) substitute known info

$$l^2 + h^2 = 10^2$$

$$h = \sqrt{10^2 - l^2} \qquad l = 6$$

$$= \sqrt{100 - 36} = 8$$

$$l = 6$$

$$\frac{dl}{dt} = 1$$

$$2l\frac{dl}{dt} + 2h\frac{dh}{dt} = 0$$

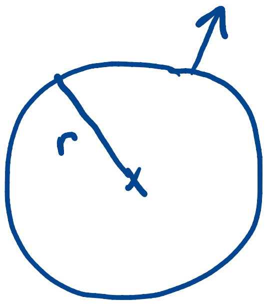
$$6 \cdot \cancel{\frac{1}{10}}\ 1 + 8 \cdot \frac{dh}{dt} = 0$$

$$\boxed{\frac{dh}{dt}} = \frac{1}{8}\left(-\frac{6}{10}\right)$$

$$= -\frac{3}{4} \cdot \cancel{\frac{1}{10}}\ 1$$

$$= \boxed{-\frac{3}{4}\ \text{ft/s}}$$

2. A pebble dropped into a calm pond, causing ripples in the form of circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the area A of the water disturbed changing?



r: radius of dist. circle, ft

t: time, sec

A: area of dist. circle, ft^2

1) $\frac{dr}{dt} = 1$ ft/sec

2) want $\frac{dA}{dt}$

3) Need a relation between $\boxed{r}$ and $\boxed{A}$.

$$\underbrace{\text{relation between}}_{A = \pi r^2}$$

4) Take a derivative

$$\frac{d}{dt} A = \frac{d}{dt} \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

5: substitute known info

$r = 4$ ft.

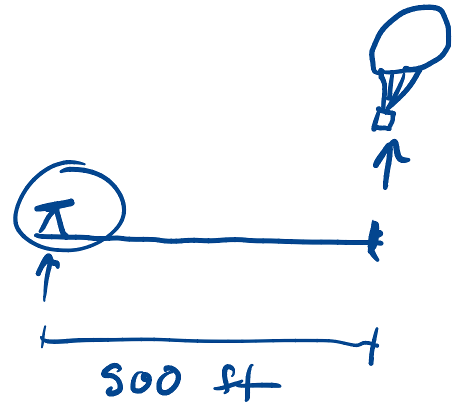
$\frac{dr}{dt} = 1$ ft/s

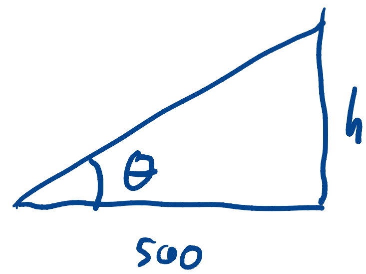
$$\frac{dA}{dt} = 2\pi \cdot 4 \cdot 1$$

$$= 8\pi$$

$$\approx \boxed{24 \text{ ft}^2/\text{sec}}$$

3. A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians/min. How fast is the balloon rising at that moment?





0) time, t, minutes

1) angle, θ, radians

know $\dfrac{d\theta}{dt} = 0.14 \dfrac{\text{rad}}{\text{min}}$

2) height, h, feet

want $\dfrac{dh}{dt} \leftarrow \dfrac{\text{feet}}{\text{min}}$

3) Need relation between θ and h.

$$\tan\theta = \frac{h}{500}$$

4) Take an implicit deriv. w.r.t t.

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\frac{h}{500}$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{dh}{dt}$$

5) Substitute

$\theta = \pi/4$

$\dfrac{d\theta}{dt} = 0.14$

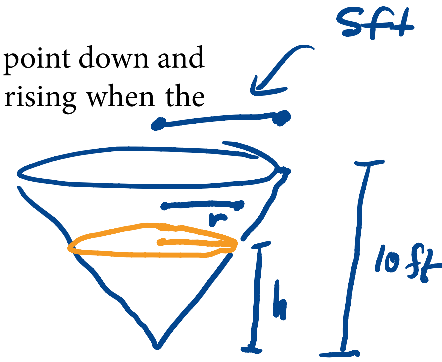
$\sec(\pi/4) = \dfrac{1}{\cos(\pi/4)} = \sqrt{2}$

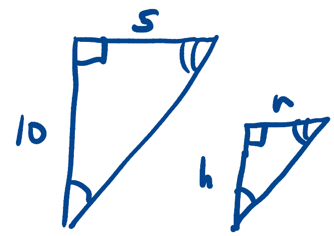
$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{dh}{dt}$$

$$(\sqrt{2})^2 \cdot 0.14 = \frac{1}{500}\frac{dh}{dt}$$

$$\frac{dh}{dt} = 500 \cdot 2 \cdot 0.14 = 140 \text{ ft/min}$$

4. Water runs into a conical tank at the rate of 9 ft³/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?





$$\frac{5}{10} = \frac{r}{h}$$

0) time, t, minutes

1) volume of water in tank, V, ft^3

$$\frac{dV}{dt} = 9 \ ft^3/min$$

2) height of water in tank, h, ft

$$\frac{dh}{dt} = ?$$

3) Need is a relation between V and h.

$$V = \frac{1}{3}\pi r^2 h$$

(is r perhaps related to h?)

$$r = \frac{1}{2} h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$= \frac{1}{12}\pi h^3$$

4) $\frac{d}{dt}V = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right)$

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt}$$

$$= \frac{\pi}{4} h^2 \frac{dh}{dt}$$

5) Substitute: $\frac{dV}{dt} = 9, \quad h = 6$

$$9 = \frac{\pi}{4} \cdot 36 \cdot \frac{dh}{dt}$$

$$9 = 9\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \approx \frac{1}{3} \text{ ft/min}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \frac{\text{ft}}{\text{min}}$$

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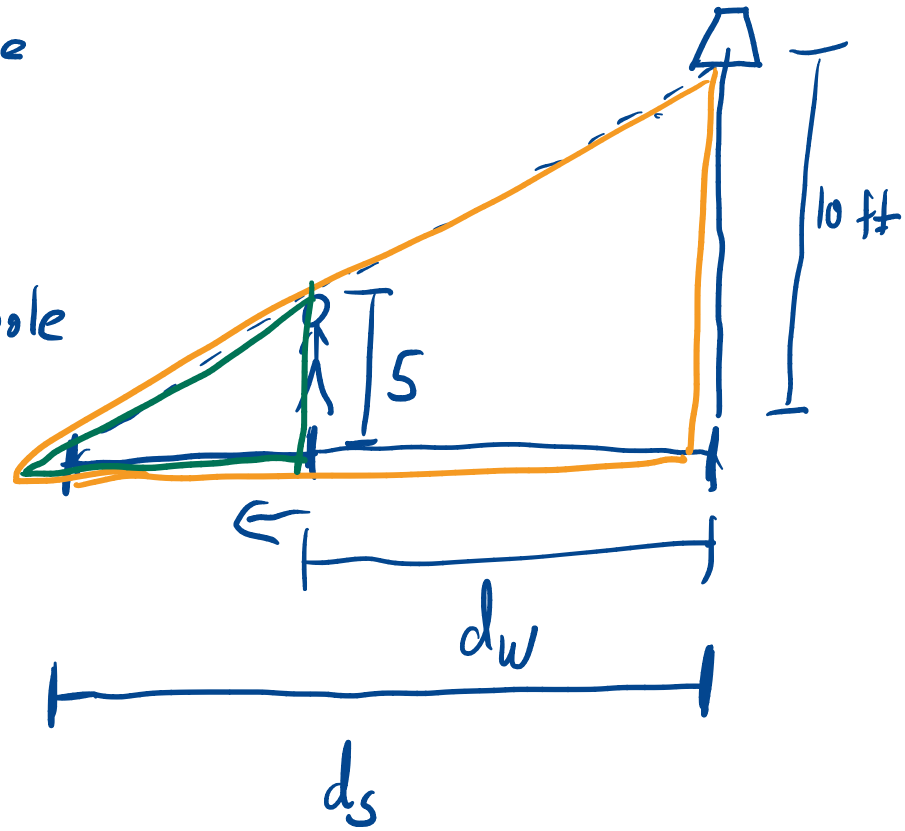
5. A street light is mounted at the top of a 10-ft-tall pole. A woman 5 ft tall walks away from the pole along a straight path at a speed of 5 ft/s. How fast is the tip of her shadow moving when she is 40 ft from the pole?

0) rates with respect to time

time, t, seconds



1) dist. between woman & pole

d_W, feet

$$\frac{d\,d_W}{dt} = 5 \text{ ft/s}$$

2) dist between shadow tip & pole

d_s, feet

$$\frac{d\,d_s}{dt} = ?$$

3) Need a relation between d_s and d_W

$$\frac{d_s}{10} = \frac{d_s - d_W}{5}$$

$$\frac{1}{2} d_s = d_s - d_W$$

$$d_s = 2d_s - 2d_W$$

$$d_s = 2\,d_W$$

$$d_s = 2\, d_w$$

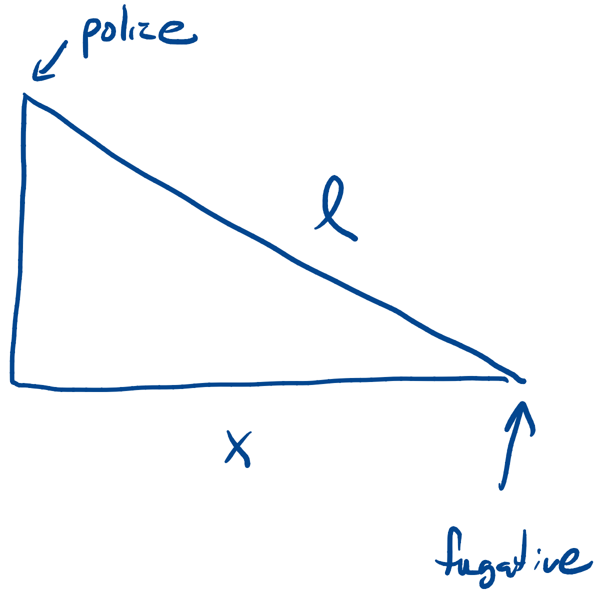
4) $$\frac{d}{dt} d_s = \frac{d}{dt} 2\, d_w$$

$$\frac{d\, d_s}{dt} = 2 \frac{d\, d_w}{dt}$$

5) Substitute: $$\frac{d\, d_w}{dt} = \frac{5 \text{ ft}}{\text{sec}}$$

$$\frac{d\, d_s}{dt} = 2 \cdot 5 = 10 \text{ ft/sec}$$

6. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine that the distance between them and the car they are chasing is increasing at a rate of 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car? [Hint: You'll need to relate *three* quantities here!]



0) time, t, seconds

1) a) dist between police + intersection y

y, miles

$$\frac{dy}{dt} = -60 \text{ mph}$$

distance is decreasing

b) dist between fugative + police

l, miles

$$\frac{dl}{dt} = 20 \text{ mph}$$

2) dist between fugatwe + intersection

y, miles

3) relation: $x^2+y^2=\ell^2$

4) $$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2\ell\frac{d\ell}{dt}$$

$$x\frac{dx}{dt}+y\frac{dy}{dt}=\ell\frac{d\ell}{dt}$$

$y=0.6$, $x=0.8$

$x^2+y^2=\ell^2 \Rightarrow \ell=1$ mile

$$0.8\frac{dx}{dt}-0.6\cdot 60=1\cdot 20$$

$$\frac{dx}{dt}=\frac{1}{0.8}\cdot\left[20+36\right]$$

$$=\boxed{70 \text{ mph}}$$