Linearization

Given a function $f(x)$, its linearization at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x - a).$$

For example, if $\boxed{f(x) = \sqrt{x}}$ and $a = 4$ then $f(4) = 2$ and $f'(4) = 1/(2\sqrt{4}) = 1/4$. So

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

The graph of the linearization is just the tangent line to the curve $y = \sqrt{x}$ at $x = 4$. So we expect that $L(x)$ is a good approximation for $\sqrt{x}$ for x near 4. The point is that computing square roots is hard work (even if your calculator makes it look easy) but computing the value of a linear function like L is easy. In fact your calculator is doing a more sophisticated generalization of the linear approximation: stay tuned in Calculus II!

→ find the equation of the tangent line at $x=4$

$x=4$, $y = f(4) = \sqrt{4} = 2$

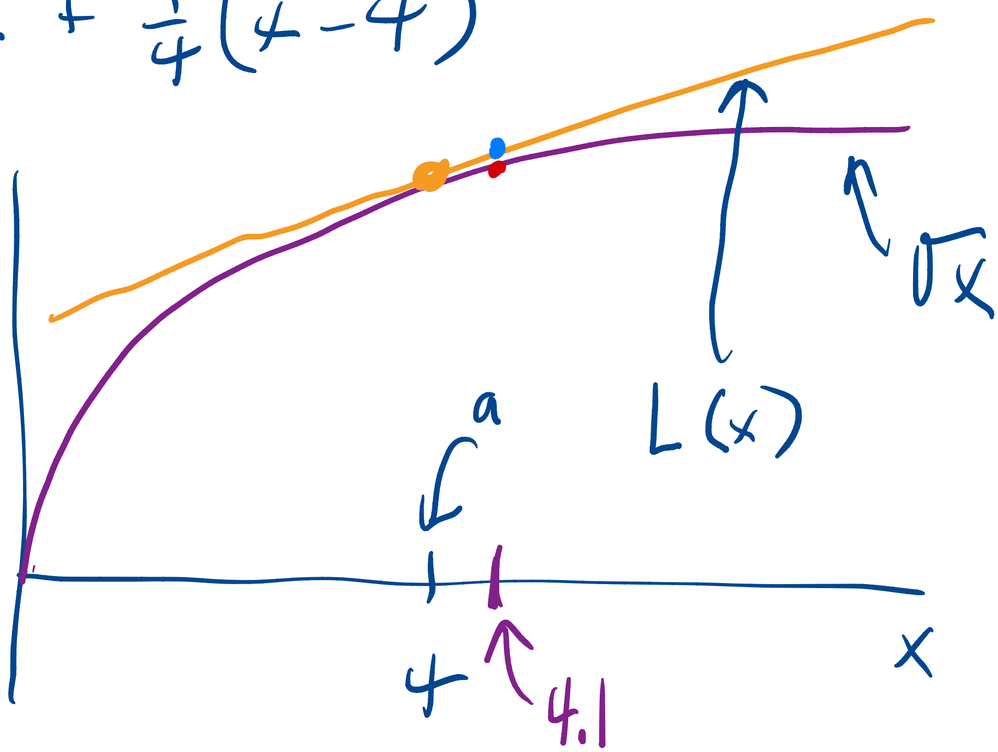
$f'(4)$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x-4)$

$y = 2 + \frac{1}{4}(x-4)$



1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$$f(x) = \sqrt{x}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$a = 4 \qquad f(a) = 2 \qquad f'(a) = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-4)$$

$L(x) \approx f(x)$ if x is near 4.

$$L(4) = 2 + \frac{1}{4}(4-4) = 2 = \sqrt{4} = f(4)$$

$$L(4.1) \approx f(4.1) = \sqrt{4.1}$$

$$L(4.1) = 2 + \frac{1}{4}(4.1-4) = 2 + \frac{1}{4}\cdot\frac{1}{10}$$

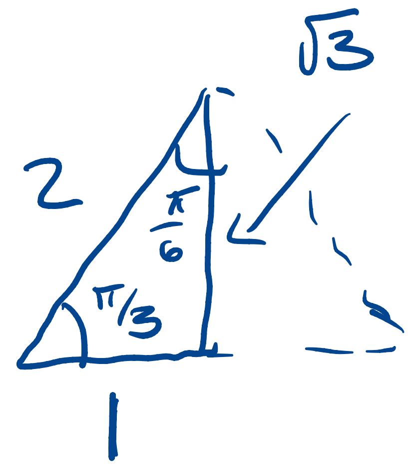
$$= 2 + \frac{1}{40} = \boxed{2.025}$$

$$\sqrt{4.1} = 2.0248\ldots$$

2. Use the linear approximation to approximate the cosine of $29° = \frac{29}{30}\frac{\pi}{6}$ radians.

$$29° \cdot \frac{2\pi}{360°} = \frac{29}{30} \cdot \frac{\pi}{6}$$

$$L(x) = f(a) + f'(a)(x-a)$$



$$f(x) = \cos(x)$$

Want to approximate $f\left(\frac{29}{30}\frac{\pi}{6}\right)$

we'll use linearization

we need: a choice $a \to \frac{\pi}{6}$

1) a is close to $\frac{29}{30}\frac{\pi}{6}$

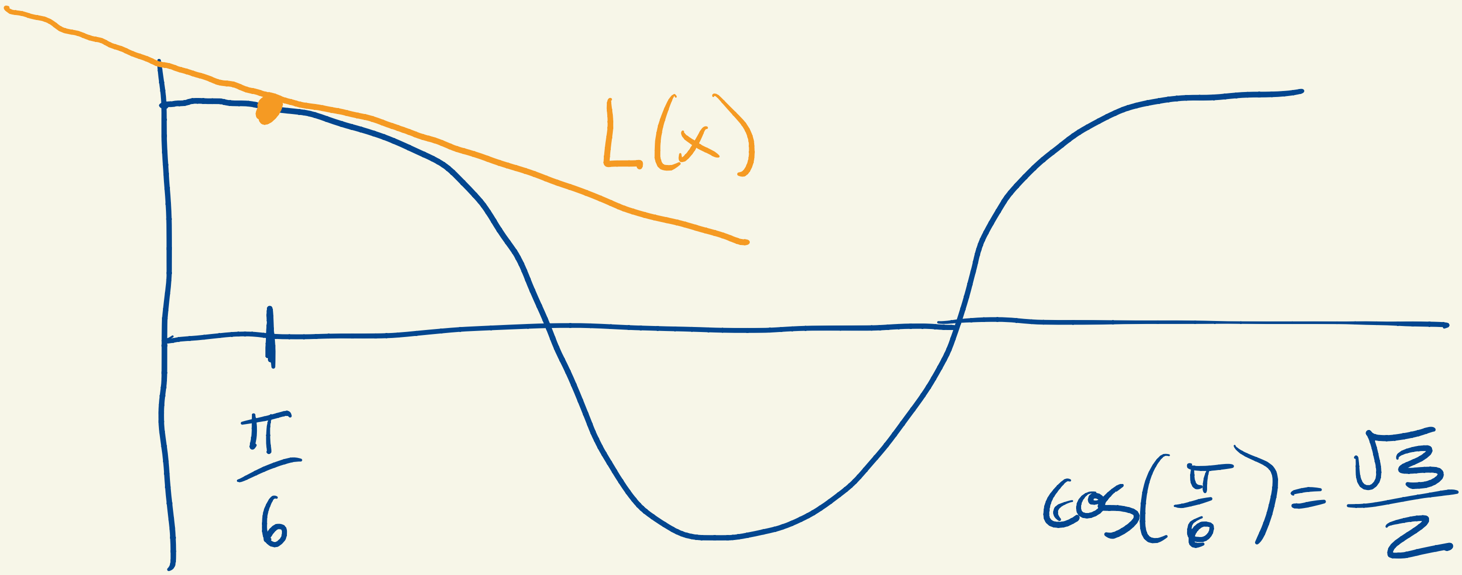
2) we can compute $f(a)$ and $f'(a)$ to obtain the linearization

$$L(x) = f(a) + f'(a)(x-a)$$

$$f\left(\tfrac{\pi}{6}\right) = \cos\left(\pi/6\right) = \frac{\sqrt{3}}{2}$$

$$f'\left(\tfrac{\pi}{6}\right) = -\sin\left(\tfrac{\pi}{6}\right) = -\frac{1}{2}$$

$$L(x) = \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(x - \frac{\pi}{6}\right)$$



$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Want: $\cos\left(\frac{29}{30}\frac{\pi}{6}\right)$

Use: $L\left(\frac{29}{30}\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{29}{30}\frac{\pi}{6} - \frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{1}{30}\frac{\pi}{6}\right)$

$\frac{\sqrt{3}}{2} = 0.866\ldots$

$= \frac{\sqrt{3}}{2} + \frac{1}{60}\frac{\pi}{6}$

$= 0.87475\ldots$

$0.87461\ldots$

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approxmate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5)$

$$L(x) = f(a) + f'(a)(x-a)$$

$f(x) = \ln(x)$

$a = 1$

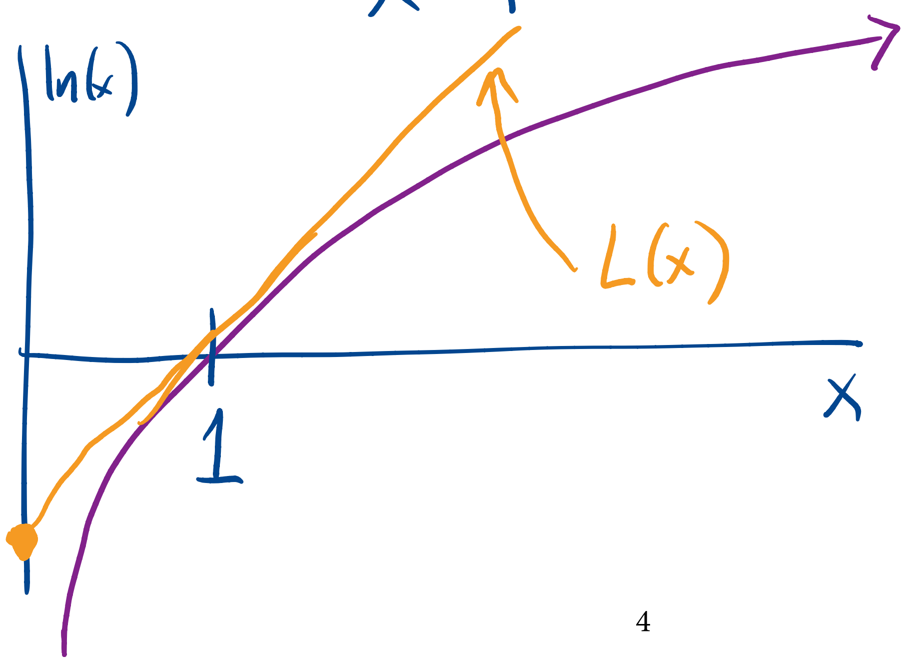
$f(a) = \ln(a)$
$= \ln(1) = 0$

$f'(a) = \ln'(a) = 1/a$

$f'(1) = \frac{1}{1} = 1$

$L(x) = x - 1$

$L(x) = 0 + 1 \cdot (x - 1)$
$= x - 1$



a) $\ln(0.5) \approx L(0.5) = L\left(\frac{1}{2}\right) = \frac{1}{2} - 1$

-0.693

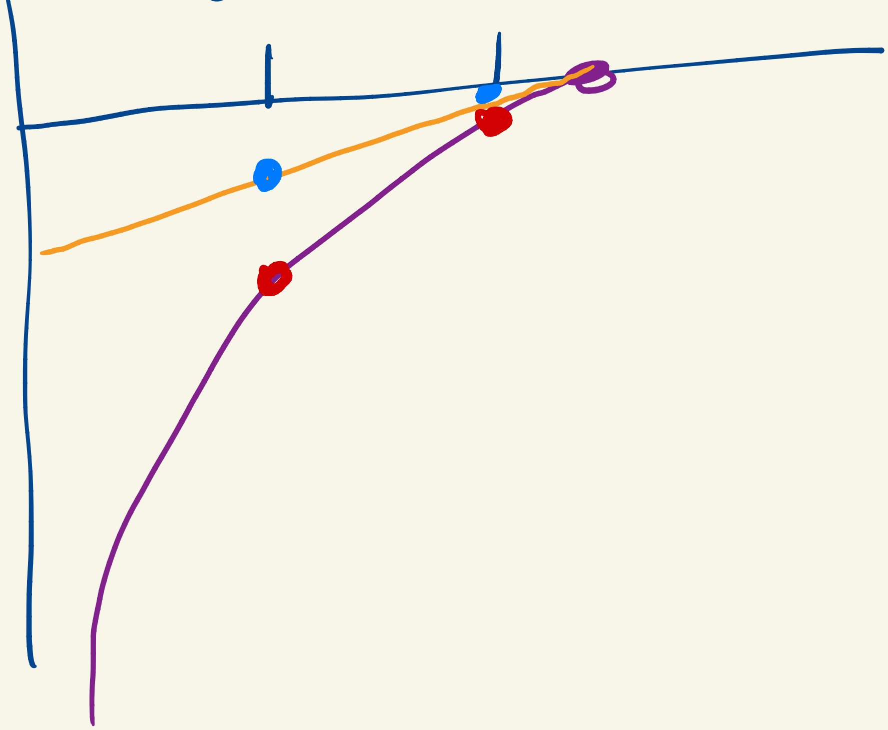
$= -\frac{1}{2} = -0.5$

b) $\ln(0.9) \approx L(0.9) = L\left(\frac{9}{10}\right) = \frac{9}{10} - 1$

-0.105

$= -\frac{1}{10}$

-0.1



4. Find the linear approximation of $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.05}$ and e^1 Compare your approximations with your calculator's.

$L(x) = f(a) + f'(a)(x-a)$

$f(x) = e^x$, $a=0 \Rightarrow f(a) = e^0 = 1$

$f'(x) = e^x$, $a=0 \Rightarrow f'(a) = e^0 = 1$

$L(x) = 1 + 1\cdot(x-0) = 1+x$

$e^{0.05} \approx L(0.05) = 1.05$

$e^{0.05} = 1.05127\cdots$

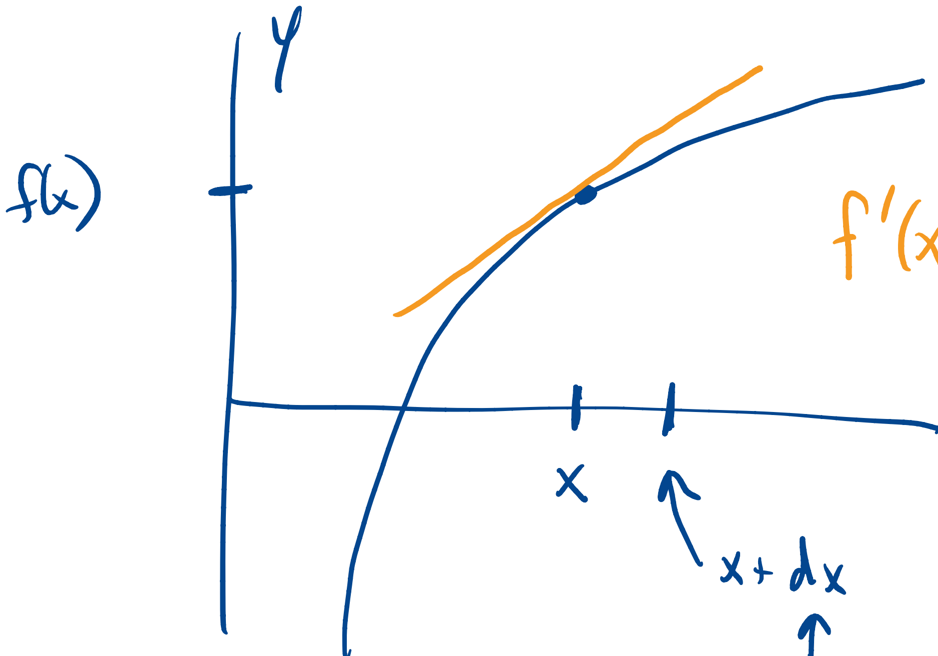
$e^1 \approx L(1) = 1+1 = 2$

$e^2 = 2.718\cdots$

Differentials Suppose we have a variable $y = f(x)$. We define its differential to be

$$dy = f'(x)dx$$

where x and dx are thought of as variables you can control. What's the point? The value of dy is an estimate of how much y changes if we change x into $x + dx$. See the graph:



$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

small change in x

How much does f(x) change when I wiggle x to x + dx

If h is small

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \qquad \underbrace{f(x+h)-f(x)}_{\Delta y} \approx f'(x)\cdot \underbrace{h}_{dx}$$

$$\Delta y \approx f'(x) \cdot dx$$

$$dy \approx \Delta y$$

$$dy = f'(x) \cdot dx$$

<u>If</u> $y = f(x)$ and we change x to $x + dx$ then y changes to approximately $y + dy$

where $dy = f'(x)\, dx$

5. A tree is growing and the radius of its trunk in centemeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

$$r(t) = 2\sqrt{t} \qquad r'(t) = 2 \cdot \frac{1}{2\sqrt{t}}$$

$$dt = \frac{1}{12} \qquad = \frac{1}{\sqrt{t}}$$

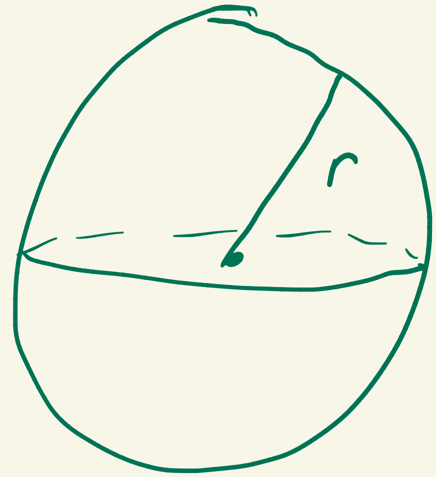
$$dr = r'(t)\, dt$$

$$dr = \frac{1}{\sqrt{t}}\, dt \qquad t = 4 \quad dt = 1/12$$

$$dr = \frac{1}{\sqrt{4}} \cdot \frac{1}{12} = \frac{1}{24} \qquad 0.0416$$

$$\Delta r = 0.0414$$

$$V(r) = \frac{4}{3}\pi r^3$$



$$\frac{dV}{dr} = 4\pi r^2 \longrightarrow dV = 4\pi r^2\, dr$$

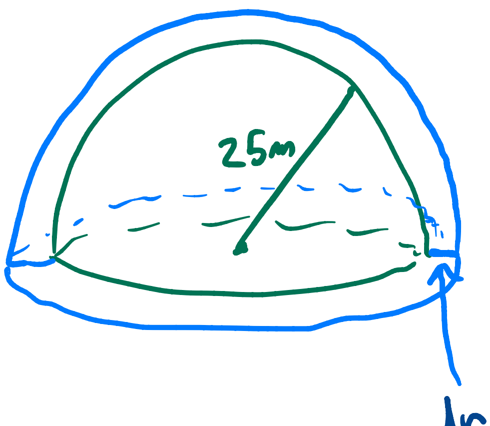
$$\frac{\Delta V}{\Delta r}$$

as $\Delta r \to 0$

If the radius is r and we change it to $r + dr$ then the volume approximately changes from V to $V + dV$.

Is a better approximation if dr is small.

6. A coat of paint of thinkness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]



$\Downarrow$

0.05 cm

$0.05\text{ cm} = \frac{0.05}{100}\text{ m}$

Volume of paint?

It's the change in volume of the hemisphere going from 25 to 25+dr m.

$$V = \frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$$

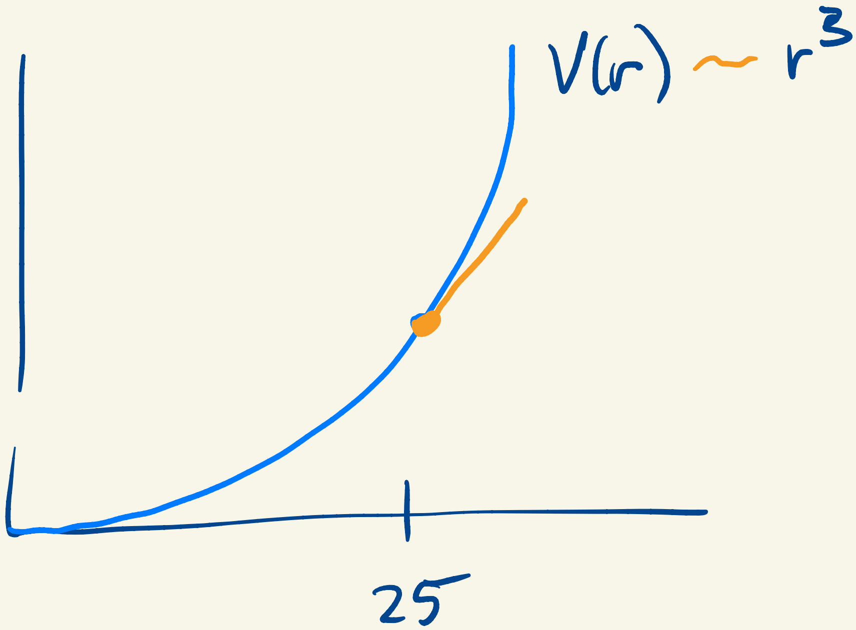
$$dV = \frac{2}{3}\pi 3r^2 \cdot dr \to 2\pi r^2$$

surface area of a hemisphere.

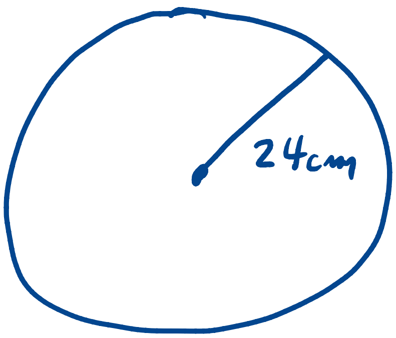
$$\boxed{r = 25\text{ m}}$$

$$\boxed{dr = \frac{0.05}{100}\text{ m}}$$

$$dV = 2\pi r^2\, dr$$
$$= 2\pi (25)^2 \cdot \left(\frac{0.05}{100}\right) \approx 1.96\text{ m}^3$$


$V(r) \sim r^3$
25

7. The radius of a disc is 24cm with an error of ±0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.



$A = \pi r^2$

$dA = 2\pi r \, dr$

$dA = 2\pi \, 24 \cdot \frac{1}{2}$

$= 75.4 \text{ cm}^2$

error in the area is $\pm 75.4 \text{ cm}^2$

absolute

$A = \pi \cdot 24^2 = 1808 \text{ cm}^2$

$\frac{dA}{A} = \frac{75.4}{1808} = 0.0416 = \pm 4.2\%$

relative error