# Math F251 — Final Exam — Fall 2018

**Name:** SOLUTIONS

**Student Id:** ____________

**Section:** ☐ F01 (Bueler) ☐ F02 (Jurkowski) ☐ F03 (Maxwell)

## Rules:

Please turn off anything that might go beep during the exam.

You have 120 minutes (2 hours) to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

When possible, place a box around your **FINAL ANSWER** for each question. If you need extra space, you can use the back sides of the pages. (Please make it clear if you have done so.) Good luck!

| Problem | Possible | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 15 | |
| 10 | 15 | |
| 11 | 10 | |
| Extra Credit | 5 | |
| Total | 140 | |

**1. (15 points)**

Differentiate the following functions.

**a.** $r(\theta) = e^{\tan\theta}$

$$r'(\theta) = e^{\tan\theta} \sec^2\theta$$

**b.** $f(x) = (x^2 - 1)\cos\left(\dfrac{\pi}{x}\right)$

$$f'(x) = 2x\cos\left(\frac{\pi}{x}\right) + (x^2-1)\sin\left(\frac{\pi}{x}\right)\frac{\pi}{x^2}$$

**c.** $g(x) = x\ln\sqrt{x} - \dfrac{x}{2}$

$$g'(x) = 1 \cdot \ln\sqrt{x} + x \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{2}$$

$$= \ln(\sqrt{x})$$

**2. (10 points)**

Find an equation of the tangent line at $x = 0$ to the curve $y = \dfrac{1+x}{1+e^x}$.

$$\frac{dy}{dx} = \frac{1\cdot(1+e^x) - (1+x)(e^x)}{(1+e^x)^2} = \frac{1-xe^x}{(1+e^x)^2}$$

$$m = \left.\frac{dy}{dx}\right|_{x=0} = \frac{1-0}{(1+1)^2} = \frac{1}{4}$$

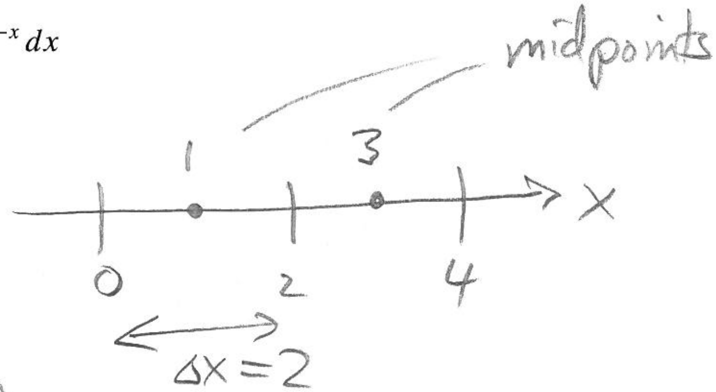
$$y\big|_{x=0} = \frac{1}{2}$$

$$\therefore \quad y - \frac{1}{2} = \frac{1}{4}(x-0)$$

**3. (10 points)**

Use the **midpoint rule** with $n = 2$ subintervals to approximate the integral:

$$\int_0^4 x\,2^{-x}\,dx$$



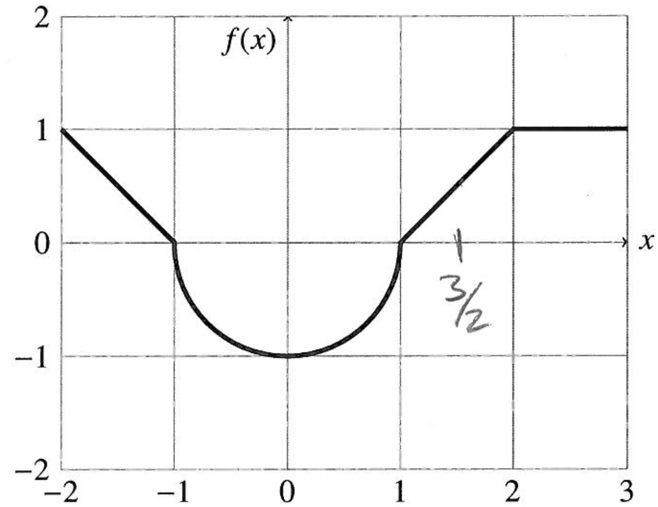

$$\int_0^4 x\,2^{-x}\,dx$$

$$\approx 2\cdot(1\cdot 2^{-1}) + 2\cdot(3\cdot 2^{-3})$$

$$= 2\left(\frac{1}{2} + \frac{3}{8}\right) = 2\left(\frac{7}{8}\right) = \frac{7}{4}$$

**4. (15 points)**

Consider the the function $f(x)$ graphed below. Between $x = -1$ and $x = 1$ the graph is a circle of radius 1.




**a.** What is the value of $f(0)$?

$f(0) = -1$

**b.** What is the value of $f'(3/2)$?

$f'(3/2) = 1$

**c.** What is the value of $\int_{-1}^{2} f(w)\,dw$?

$$\int_{-1}^{2} f(w)\,dw = \int_{-1}^{1} f(w)\,dw + \int_{1}^{2} f(w)\,dw$$
$$= -\frac{\pi}{2} + \frac{1}{2}$$

**d.** Let $g(x) = \int_{-1}^{x} f(w)\,dw$. What is the value of $g(-2)$?

$$g(-2) = \int_{-1}^{-2} f(w)\,dw = -\int_{-2}^{-1} f(w)\,dw = -\frac{1}{2}$$

**e.** For the function $g(x)$ from part **d.**, what is the value of $g'(0)$?

by FTCI, $g'(x) = f(x)$ so $g'(0) = f(0) = -1$

**5. (15 points)**

The temperature of an oven in °F is

$$T(t) = 150 + 30t^2$$

for $t = 0$ to $t = 3$ minutes.

**a.** Find the average rate of change of temperature in the oven from time $t = 0$ to $t = 3$. Include units in your answer.

$$\frac{T(3) - T(0)}{3 - 0} = \frac{(\cancel{150} + 30 \cdot 3^2) - (\cancel{150} + 0)}{3}$$

$$= 90 \ \frac{°F}{\text{minute}}$$

**b.** It is easy to compute that $T'(2) = 120$. What does this mean in everyday language? (Be sure to include units in your answer.)

this is the (instantaneous) rate of change of temperature at $t=2$ minutes; the temperature is increasing by 120°F per minute

**c.** Using the limit definition of the derivative, compute $T'(1)$. (No credit will be granted for using other methods to compute the derivative.)

$$T'(1) = \lim_{h \to 0} \frac{T(1+h) - T(1)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{150} + 30(1+h)^2) - (\cancel{150} + 30 \cdot 1^2)}{h}$$

$$= \lim_{h \to 0} \frac{30(\cancel{1} + 2h + h^2 - \cancel{1})}{h} = \lim_{h \to 0} 30(2+h)$$

$$= 60 \ \frac{°F}{\text{min}}$$

**6. (15 points)**

Evaluate the integrals. For full credit, include a constant of integration whenever one would be justified.

**a.** $\displaystyle\int \left(e^{-2t} + t^{2/5} + \frac{1}{t}\right) dt$

$$= -\frac{e^{-2t}}{2} + \frac{5}{7}t^{7/5} + \ln|t| + C$$

**b.** $\displaystyle\int \left(x\cos(x^2) - \frac{1}{x^2+1}\right) dx$

$$= \int x\cos(x^2)\,dx - \int \frac{dx}{1+x^2}$$

$$= \int \cos(u)\,\frac{du}{2} - \arctan x + C$$

$u = x^2$

$\frac{du}{2} = x\,dx$

$$= \frac{1}{2}\sin u - \arctan x + C$$

$$= \frac{1}{2}\sin(x^2) - \arctan x + C$$

**c.** $\displaystyle\int_e^{e^4} \frac{\sqrt{\ln x}}{x}\,dx$

$$= \int_1^4 \sqrt{u}\,du$$

$u = \ln x$

$du = \frac{dx}{x}$

$$= \frac{2}{3}u^{3/2}\Big]_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1)$$

$$= \frac{14}{3}$$

**7. (10 points)**

Find the limit or show that it does not exist.

**a.** $\lim_{x\to\infty} \frac{\sqrt{x^2+3}}{7x} = \lim_{x\to\infty} \frac{\sqrt{x^2+3}\cdot\frac{1}{x}}{7x\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{3}{x^2}}}{7} = \frac{1}{7}$

**b.** $\lim_{t\to 1} \frac{t^8-1}{t^5-1} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{t\to 1} \frac{8t^7}{5t^4} = \lim_{t\to 1} \frac{8}{5}t^3 = \frac{8}{5}$

**8. (10 points)**

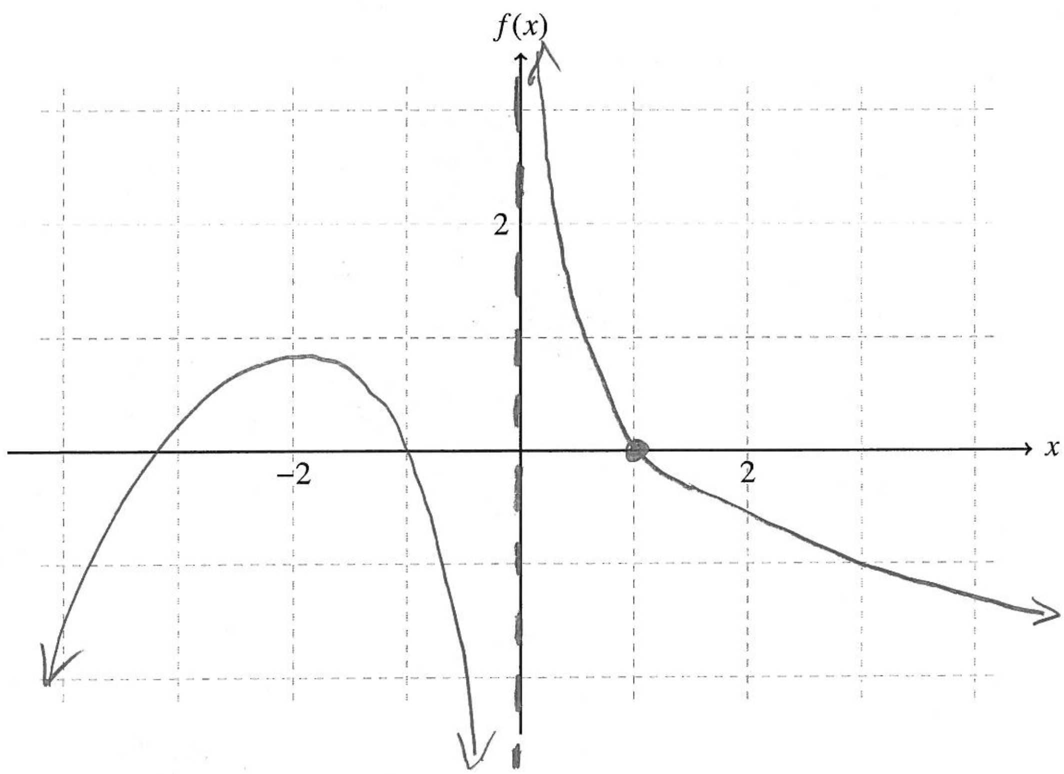
On the axes below, sketch the graph of a function that satisfies **all** of the given conditions:

**a.** $f(1) = 0$,

**b.** $f'(x) > 0$ if $x < -2$ and $f'(x) < 0$ if $x > -2$,

**c.** $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$,

**d.** there is a vertical asymptote at $x = 0$.



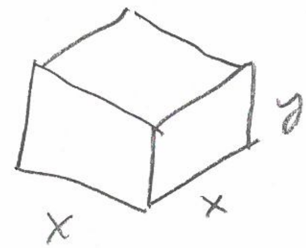

$f'$

$f''$

**9. (15 points)**

A box has a square base and an open top. The base material costs 3¢ per square inch and the sides cost 2¢ per square inch. Suppose the width of the base of the box is *x* inches and its height is *y* inches.

**a.** What is the cost of the bottom of the box?




$3x^2$ ¢

**b.** What is the total cost of the box?

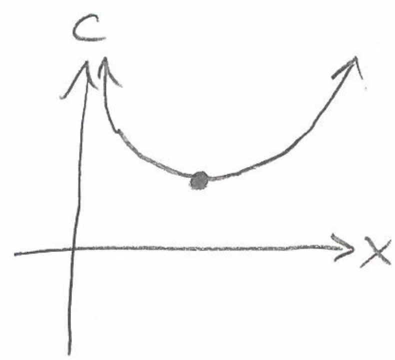
$C = 3x^2 + 4(2xy) = 3x^2 + 8xy$ ¢

**c.** Suppose the box must have a volume of 30 cubic inches. Find the dimensions (both *x* and *y*) of the box which minimizes the cost.

$30 = x^2 y \quad \therefore \quad y = \dfrac{30}{x^2}$

$C(x) = 3x^2 + 8x\left(\dfrac{30}{x^2}\right) = 3x^2 + \dfrac{240}{x}$

$C'(x) = 6x - \dfrac{240}{x^2} = 0$




$6x = \dfrac{240}{x^2}$

$x^3 = 40$

$x = (40)^{1/3}$ inches

$y = 30(40)^{-2/3}$ inches

**10. (15 points)**

Water flows from a tank at a rate of $r(t) = 3t^2 - t^3$ liters per minute from $t = 0$ to $t = 3$ minutes.

**a.** Compute $r(0)$, $r(1)$ and $r(3)$, and explain these quantities mean in everyday language. Your answer should include units.

$$\left.\begin{aligned} r(0) &= 0 \\ r(1) &= 3-1=2 \\ r(3) &= 27-27=0 \end{aligned}\right\} \quad \text{units } \frac{\text{liters}}{\text{minute}}$$

flow rates at times $t=0,1,3$ minutes

**b.** Compute the total amount of water that drains from the tank from time $t = 0$ to $t = 3$.

$$\int_0^3 r(t)\,dt = \int_0^3 3t^2 - t^3\,dt = t^3 - \frac{t^4}{4}\Big]_0^3$$

$$= 27 - \frac{81}{4} = \frac{108-81}{4} = \frac{27}{4} \text{ liters}$$

**c.** At what time is the rate of flow at a maximum? (Only consider $t$ in the interval $[0, 3]$.)

$r'(t) = 6t - 3t^2$

$6t - 3t^2 = 0$

$t = 0$ ✓ $\qquad 6 - 3t = 0$

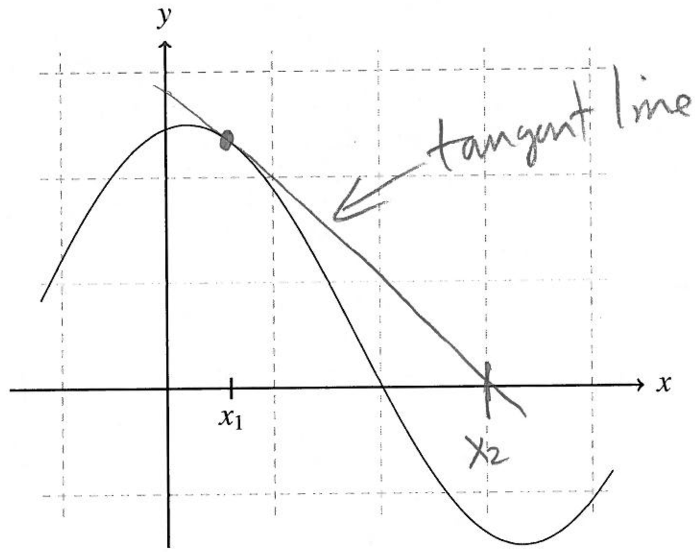
$t = 2$ minutes

| $t$ | $r(t)$ |
|---|---|
| 0 | 0 |
| 2 | 4 |
| 3 | 0 |

abs. max

**11. (10 points)**

**a.** A generic graph $y = f(x)$ is shown and a first approximation $x_1$ is indicated. Show, by adding to the sketch, how Newton's method would find the next approximation $x_2$.




**b.** For the equation $\underbrace{x^3 - 4x + 2}_{f(x)} = 0$ and the value $x_1 = -2$, compute $x_2$ from Newton's method.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 4x_1 + 2}{3x_1^2 - 4}$$

$$= (-2) - \frac{\cancel{-8} + \cancel{8} + 2}{12 - 4} = -2 - \frac{1}{4} = -\frac{9}{4}$$

**12. (Extra Credit: 5 points)**

Find **and simplify** the derivative of the function:

$$h(x) = \int_1^{e^x} \ln t \, dt$$

Explain your steps.

chain rule

$$h'(x) = \ln(e^x) \cdot e^x = x e^x$$

FTCI

$(\ln e^a) = a$