Antiderivatives

Given a rate of change,

can you construct the original function?

I give you the rate at which water is draining from a tank. Can you reconstruct the amount of water in the tank?

Def: An <u>antiderivative</u> of a function

$f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

Find an antiderivative of $f(x) = 0$

$$F'(x) = \underbrace{0}_{f(x)} \quad \text{(everywhere)}$$

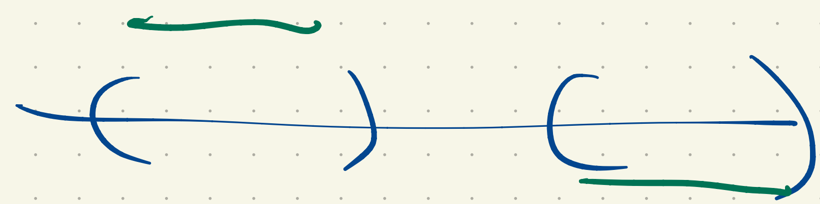
$$F(x) = 1 \quad \text{(everywhere)}$$

$$F(x) = \sqrt{\pi} \quad \text{(everywhere)}$$

$F(x) = C$ for any constant C.

Are there any others?

If $F(x)$ is defined on an interval and $F'(x) = 0$ (everywhere) then $F(x)$ is constant.



Follows from Mean Value Theorem.

e.g. Find an antiderivative of x^2.

Want $F(x)$ with $F'(x) = x^2$.

$$F(x) = \frac{1}{3}x^3$$

$$F'(x) = \frac{1}{3}3x^2 = x^2$$

Or, $F(x) = \frac{1}{3}x^3 + 9$.

$$F'(x) = \frac{1}{3}x^3 + 0 = \frac{1}{3}x^3$$

Or, $F(x) = \frac{1}{3}x^3 + C$ for

any constant C.

If $G(x)$ satisfies $G'(x) = x^2$

then $\frac{d}{dx}\left[G(x) - \frac{x^3}{3}\right] = x^2 - x^2 = 0$

$$G(x) - \frac{x^3}{3} = C$$

$$G(x) = \frac{x^3}{3} + C$$

(so long as $G(x)$ is defined on an interval).

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e.g. Find all antiderivatives of $\sin(x)$.

$$F(x) = -\cos(x)$$

$$F'(x) = -\frac{d}{dx}\cos(x) = -(-\sin(x)) = \sin(x)$$

$$\underbrace{F(x) = -\cos(x) + C}_{\text{all antiderivatives}}$$

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e.g. Find an antiderivative(s) of

all

$$\underbrace{x^2}_{\frac{x^3}{3} + C_1} + 7\underbrace{\sin(x)}_{-\cos(x) + C_2}$$

$$F(x) = \frac{1}{3}x^3 - 7\cos(x) + \overbrace{C}^{C_1 + 7C_2}$$

Find an antiderivative of $-50e^{-t}$

$$F(t) = 50e^{-t}$$

Water is draining from a tank at a rate of $50e^{-t}$ liters per minute. What is the volume of water in the tank at time t?

$V(t) \rightarrow$ volume of water in tank at time t. (liters)

$$V'(t) = -50e^{-t}$$

$V(t)$ is an antiderivative of $-50e^{-t}$

$$\underbrace{V(t) = 50e^{-t} + C}$$

$$V'(t) = 50(-1)e^{-t} + 0$$

$$= -50e^{-t}$$

If we know $V(0) = 300\,\ell$

we can reconstruct $V(t)$ for all t.

$$V(t) = 50e^{-t} + C$$

$$V(0) = 300$$

$$V(0) = 50e^{-0} + C = 50 + C$$

$$50 + C = 300$$

$$C = 250$$

$$\boxed{V(t) = 50e^{-t} + 250 \qquad V(0) = 300}\ (\ell)$$

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty}\left(50e^{-t} + 250\right)$$

$$= 50\cdot 0 + 250$$

$$= 250\ (\ell)$$

Eventually, the tank contains $250\,\ell$.