Applied Optimization

Looking to maximize or minimize a quantity of interest $Q(x)$

Tools

1) Closed Interval Method

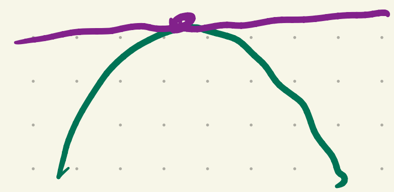
$Q(x)$ is defined on $[a, b]$

To maximize $Q(x)$

1) find the critical points:

$$Q'(x) = 0$$

or DNE



2) Look at: a) crit points

b) end points

2) Concavity method

Q(x) is defined on an interval and has just one critical point

concave down

$Q''(x) < 0$ everywhere

Q''s derivative is negative

$$\left.\begin{array}{l} Q'(c) = 0 \\ Q''(x) < 0 \text{ everywhere} \end{array}\right] \Rightarrow \text{absolute max at } c$$

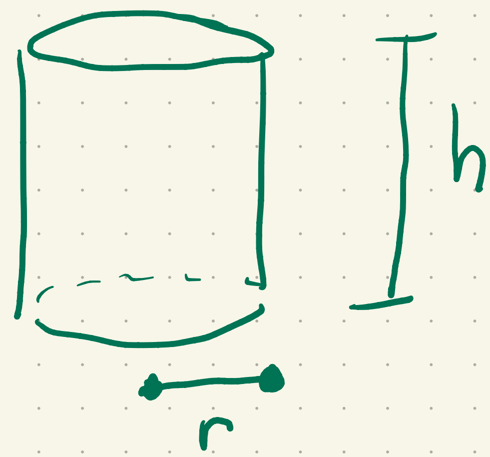
$$\left.\begin{array}{l} Q'(c) = 0 \\ Q''(x) > 0 \text{ everywhere} \end{array}\right] \Rightarrow \text{abs. minimum at } c$$

Example:

Suppose a can has fixed volume V.

What dimensions of the can minimize the surface area?

material costs



1) Read the problem.

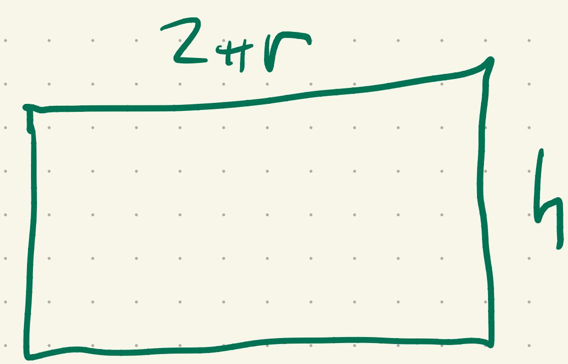
2) Draw a picture and label!

3) Introduce the quantity Q to optimize and write it in terms of other variables.

top surface area: πr^2

bottom surf area: πr^2

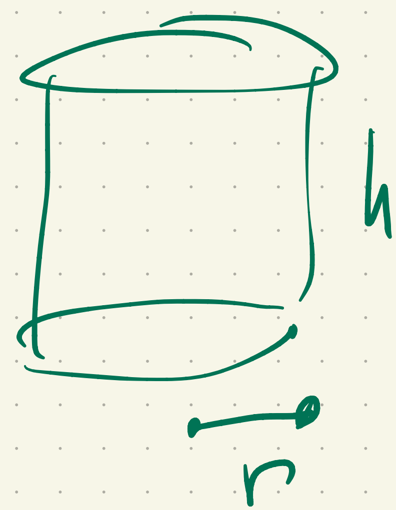
wall surf area: $2\pi r h$



$$A = 2\pi r^2 + 2\pi rh$$

4) Use relations to reduce the number of independent variables down to just one.

We'll use V.



$10 in^3$

$$V = h\pi r^2$$

$$h = \frac{V}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2V}{r}$$

$$A = 2\pi r^2 + \frac{2V}{r}$$ ⟵ minimize this for all $r > 0$

5) Apply calculus: look for $\frac{dA}{dr} = 0$

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2} = 0$$

$$4\pi r = \frac{2V}{r^2}$$

$$4\pi r^3 = 2V$$

$$r^3 = \frac{V}{2\pi}$$

$$r = \left(\frac{V}{2\pi}\right)^{1/3}$$

just one crit. point!

$$\underbrace{\frac{dA}{dr}}_{A'} = 4\pi r - \frac{2V}{r^2}$$

$$A'' = 4\pi + \frac{4V}{r^3}$$

> 0 for all $r > 0$

$A'' > 0$ everywhere

$A'' < 0$ everywhere

$r > 0$

Concave up everywhere!

Final answer:

dimensions

$$r = \left(\frac{V}{2\pi}\right)^{1/3}$$

$$h = \frac{V}{\pi} r^{-2}$$

$$= \frac{V}{\pi}\left(\frac{V}{2\pi}\right)^{-2/3}$$

$$= \frac{V}{\pi}\left(\frac{2\pi}{V}\right)^{2/3}$$

$$= V^{1/3}\ \pi^{-1}(2\pi)^{2/3}$$

$$h = \frac{V}{\pi r^2}$$

$$\pi r^2 h = V$$