Exponential Function

Examples	Rules
$5^6 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ $= (5 \cdot 5 \cdot 5 \cdot 5) \cdot (5 \cdot 5)$ $= 5^4 \cdot 5^2$	$(r > 0,\ a, b \in \mathbb{R})$ $r^a r^b = r^{a+b}$
$(5^2)^3 = 5^2 \cdot 5^2 \cdot 5^2$ $= 5^{2+2+2}$ $= 5^6$	$(r^a)^b = r^{ab}$

$$(2\cdot 7)^3 = (2\cdot 7)(2\cdot 7)(2\cdot 7)$$
$$= (2\cdot 2\cdot 2)(7\cdot 7\cdot 7)$$
$$= 2^3 7^3$$

$(r, s > 0)$

$$(rs)^a = r^a s^a$$

Consequences

$$r^0 = 1$$

$$r = r^1 = r^{1+0} = r^1 \cdot r^0 = r \cdot r^0$$

$$r = r \cdot r^0$$

$$1 = r^0$$

$$r^{-1} = \frac{1}{r} \qquad 1 = r^0 = r^{1+(-1)} = r \cdot r^{-1}$$

$r^{-1} = 1/r$

Two related kinds of functions:

a) $f(x) = x^3$ (power functions $\sqrt{x}$, $x^{2/3}$, x^4)

b) $f(x) = 3^x$ exponential functions

Exponential functions describe doubling and halving phenomena in the real world

e.g. A population of caribou grows at 10% per year and starts with 1000 animals at time $t=0$ years.

Claim: $P(t) = 1000\,(1.1)^t$

Did this work?

$$P(0) = 1000 \cdot (1.1)^0 = 1000 \cdot 1 = 1000$$

$$P(1) = 1000 \cdot (1.1)^1 = 1000\left(1 + \frac{1}{10}\right)$$

$$= 1000 + 1000 \cdot \frac{1}{10}$$

$$= 1000 + 100$$

$= 1100$

(grew by 10%)

$$P(2) = 1000\,(1.1)^2 = \left[1000 \cdot (1.1)\right] \cdot (1.1)$$

$$= 1100 \cdot (1.1)$$

$$= 1100 \cdot \left(1 + \frac{1}{10}\right)$$

$$= 1100 + 1100 \cdot \frac{1}{10}$$

$$= 1100 + 110$$

$= 1210$

↑ 10% more than 1100

What's the population after 1 year 6 months?

$$P(t) = 1000\,(1.1)^t$$

$$P(1.5) = 1000\,(1.1)^{1.5} \approx 1153.689\ldots$$

Where is the doubling?

Consider: $f(x) = 2^x$

$f(0) = 2^0 = 1$

$f(1) = 2^1 = 2$ } double

$f(2) = 2^2 = 4$ } double

Consider $f(x) = 2^{x/3}$

$x \to 3$ ↓

$f(0) = 2^{0/3} = 1$

$f(3) = 2$ } doubled

} doubled

$\times 1\nearrow 3$ $\quad f(6) = 2^{6/3} = 2^2 = 4$

$\times 1\nearrow 3$ $\quad f(9) = 8$ $\quad$ doubled

$$f(x) = 2^{x/3} = (2^{1/3})^x = (1.26)^x$$

r^x

Doubling includes halving.

$$f(x) = 2^{-x}$$

x	0	1	2	3
$f(x)$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

2^{-1}