Let's look at a more general method

$$u_{i,j+1} = A u_{i+1,j} + B u_{i,j} + C u_{i-1,j}$$

upwind: $A = 0$
$B = 1-\lambda$ $\qquad \lambda = \frac{ka}{h}$
$C = \lambda$

$$u(x,t+k) = u(x,t) + u_t(x,t)k + \frac{1}{2}u_{tt}(x,t)k^2 + \frac{1}{6}u_{ttt}(x,t)k^3 + O(k^4)$$

$$u_t = -a\,u_x$$
$$u_{tt} = \partial_t(a u_x)$$
$$= \partial_x(a u_t)$$
$$= a\partial_x(a u_x) = a^2 u_{xx}$$
$$u_{ttt} = -a^3 u_{xxx}$$

So

$$u(x,t+k) = u(x,t) - \frac{ak}{h} u_x(x,t)\,h + \frac{1}{2}\frac{(ak)^2}{h^2} u_{xx}(x,t)\,h^2$$

$$- \frac{1}{6}\frac{(ak)^3}{h^3} u_{xxx}(x,t)\,h^3 + O(k^4)$$

$$u(x\pm h,t) = u(x,t) \pm u_x h + u_{xx}\frac{h^2}{2} \pm \frac{u_{xxx} h^3}{6} + O(h^4)$$

$$c_{i,j} = u(x_i,t_j+k) - \left[A\,u(x_i+h,t_j) + B\,u(x_i,t_j) + C\,u(x_i-h,t_j)\right]$$

$$= \frac{1}{k} u(x_i,t_j)\left[1-(A+B+C)\right]$$

$$\frac{ak}{h} = \lambda$$

$$+ \left[-\lambda - A + C\right] u_x \frac{h}{k}$$

$$+ \left[\frac{\lambda^2}{2} - \frac{A}{2} - \frac{C}{2}\right] u_{xx} \frac{h^2}{k}$$

$$+ \left[-\lambda^3/6 - \frac{A}{6} + \frac{C}{6}\right] u_{xxx} \frac{h^3}{k}$$

$$+ O(k^4+h^4)\frac{1}{k} = O\left(k^3 + \frac{h^4}{k}\right)$$

needs better treatment

Consistency: $A + B + C = 1$

$$A - C = -\lambda$$

upwind: $A = 0$ $\quad C = \lambda$ ✓

$$B = 1 - \lambda$$

Maximize accuracy:

$$A + B + C = 1$$

$$A - C = -\lambda$$

$$A + C = \lambda^2$$

$$\Rightarrow \quad A = (\lambda^2 - \lambda)/2$$

$$C = (\lambda^2 + \lambda)/2$$

$$A + C = \lambda^2$$

$$B = 1 - \lambda^2$$

Next coeff: $\dfrac{C - A - \lambda^3}{6} = \dfrac{(\lambda - \lambda^3)}{6}$

$$\left(\frac{\lambda - \lambda^3}{6}\right)\frac{h^3}{k} u_{xxx} = \frac{1-\lambda^2}{6} \frac{\lambda h}{k} h^2 u_{xxx}$$

$$= a\left(\frac{1-\lambda^2}{6}\right) h^2 u_{xxx} \quad \neq 0 \text{ in general}$$

$$= (ah^2 - a^2k^2)/6 \, u_{xxx}$$

This method is known as Lax Wendroff '60.

It has $O(h^2) + O(k^2)$ order of accuracy

at least for smooth solutions.

CFL condition

so $\frac{1}{a} \geq 1 \quad a > 0$

$\frac{1}{a} \leq -1 \quad a < 0$

$-1 \leq a \leq 1 \Rightarrow |a| \leq 1$

$A \neq 0$ (not upwind)
is a problem if $a > 0$

We'll need a boundary condition
at far end

We can use upwinding there (but need to worry
about the error we
introduce).

$$u_{i,j+1} = -\frac{1}{2}\lambda(1-\lambda)u_{i+1,j} + (1-\lambda)^2 u_{i,j} + \frac{1}{2}\lambda(1+\lambda)u_{i-1,j}$$

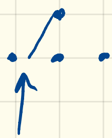
$$= u_{i,j} - \frac{1}{2}\lambda\left(u_{i+1} - u_{i-1}\right) + \frac{1}{2}\lambda^2\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$= u_{i,j} - \frac{ak}{2h}\left(u_{i+1,j} - u_{i-1,j}\right) + \frac{(ak)^2}{2h^2}\left(u_{i+1} - 2u_i + u_{i-1}\right)$$

$$u(x,t+k) = u(x,t) + u_t k + \frac{1}{2}u_{tt}k^2 + \cdots$$

$$= u - a u_x k + \frac{1}{2}a^2 u_{xx} k^2$$

centred diff   centred diff



do quadratic interpolation, and grab value with characteristic with slope $\frac{1}{a}$.

All this would be useless if the method were unstable.

Fourier method again.

Ignore boundary

$$u_{i,j} = K^{i} e^{rjhI}$$

$$Ke^{rjhI} = Ae^{r(j+1)hI} + Be^{rjhI} + Ce^{-rjhI}$$

$$K = Ae^{rhI} + B + Ce^{-r\cdot hI}$$

$$= (1-\lambda^2) + \frac{\lambda^2-\lambda}{2} e^{rhI} + \frac{\lambda^2+\lambda}{2} e^{-rhI}$$

$$= (1-\lambda^2) + \lambda^2 \cos\theta - \lambda \sin\theta \, I$$

$$= 1 + 2\lambda^2\left(\frac{\cos\theta - 1}{2}\right) - \lambda I \sin\theta$$

So

$$|K|^2 = \left(1 + \lambda^2\left[\cos\theta - 1\right]\right)^2 + \lambda^2 \sin^2\theta$$

$$= 1 + 2\lambda^2(\cos\theta - 1) + \lambda^4(\cos\theta - 1)^2 + \lambda^2(1 - \cos^2\theta)$$

$$= 1 + \lambda^2\left[2\cos\theta - 1 - \cos^2\theta + \lambda^2(\cos\theta - 1)^2\right]$$

$$= 1 + \lambda^2\left[-(1-\cos\theta)^2 + \lambda^2(\cos\theta - 1)^2\right]$$

$$= 1 + \lambda^2\left[\lambda^2 - 1\right](\cos\theta - 1)^2$$

$$= 1 - 4\lambda^2(1-\lambda^2)\sin^4\frac{\theta}{2} \qquad \theta = \frac{2\pi s h}{L}$$

$$\approx 1 - 4\lambda^2(1-\lambda^2)\left(\frac{\theta}{2}\right)^4$$

θ small $\Rightarrow$ well resolved

Compare:

$$|K|^2 = 1 - 4\lambda(1-\lambda)\sin^2\frac{\theta}{2}$$

$$\approx 1 - 4\lambda(1-\lambda)\left(\frac{\theta}{2}\right)^2$$

$\cdots$

U^{n+1} C A B

C A B

C A B

$\ddots$

[ ]

bumps

upward $h = 0.05$

.03

.01

lw $h = 0.05$

0.03

0.01

both : 0.05 → discuss log

blade

upward $h=0.05$ $\lambda=0.8$

$\lambda=0.5$

↑

why the echo decay?

lw $h=0.05$, $\lambda=0.5$

$\lambda=0.8$

$h=0.01$ $\lambda=0.8$

Upwind: $\quad \tau_{ij} = -\frac{1}{2} a h (1-\lambda) u_{xx} + \cdots$

LW $\quad \tau_{ij} = \frac{1}{6} (a h^2 - a^2 k^2) u_{xxx}$

$$u_t - a u_x = \frac{1}{2} a h (1-\lambda) u_{xx}$$

vs

$$u_t - a u_x = \frac{1}{6} (a h^2 - a^2 k^2) u_{xxx}$$