Last class:

showed convergence of upwind method for $u_t + a u_x = 0$

assuming $0 \le \underbrace{\frac{ak}{h}}_{\lambda} \le 1$

and: u_{tt}, u_{xx} exist and are continuous (which fails in many cases of interest).

Proof was via a maximum principle idea.

Alternative perspective on how CFL is playing a role

$$\vec{u}_{j+1} = B \vec{u}_j$$

$$B = I - \lambda A$$

MOL to $\vec{u}' = -\frac{a}{h} A \vec{u}$

$$A = \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & -1 & \ddots & \\ 0 & & -1 & 1 \end{bmatrix} \longmapsto A_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ & & \text{same} & & \\ & & & & \end{bmatrix}$$

corresponds to periodic instead of Dirichlet BC's

Eigenvectors of A_ρ

$$u_j - u_{j-1} = \kappa u_j \qquad \text{except} \qquad u_0 - u_N = \rho u_0$$

try $\quad u_j = e^{Irx_j}$

$$e^{Irx_j}\left(1 - e^{-rIh}\right) = \kappa e^{Irx_j}$$

So $\quad \kappa = 1 - e^{-rhI}$

But this ignores $\quad u_0 - u_N = \kappa u_0$

$$1 - e^{r\ell I} = \kappa = 1 - e^{-rhI}$$

$$\Rightarrow \quad e^{Ir\ell} = e^{rhI}$$

$$\Rightarrow \quad e^{r(\ell+h)I} = 1$$

$$\Rightarrow \quad r(\ell+h) = 2\pi k$$

$$r = \frac{2\pi k}{\underbrace{\ell+h}_{L}}$$

In fact $u_i = e^{Irx_i}$ is almost an eigenvector for any choice of r; only failure is at

$$u_0 - u_N = \left(1 - e^{IrL}\right) u_0$$

$\uparrow$

bounded if r is real.

~~[illegible]~~

eigenvalues of A_P are

$$1 - e^{-rhI} \qquad r = \frac{2\pi k}{L}$$

We applied method of lines to

$$\vec{u}' = -\frac{a}{h} A_p \vec{u}$$

all eigenvalues have

real part $-\frac{a}{h}(1-\cos\theta)$

$\leq 0.$

$\theta = \nu h$

So we'd need

$$\left| \frac{ka}{h}\left(-1+e^{I\theta}\right) + 1 \right| \leq 1$$

for absolute stability to hold.

(finally! complex e.g's!)

$$\left| \lambda\left(-1+e^{-I\theta}\right)+1 \right|^2 = \left| 1-\lambda+\lambda\cos\theta - \lambda\sin\theta\, I \right|^2$$

$$= 1-2\lambda+\lambda^2+\lambda^2$$

$$+2(1-\lambda)\lambda\cos\theta$$

$$\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$$

$$= 1 + 2\lambda(1-\lambda) - 2\lambda(1-\lambda)\cos\theta$$

$$= 1 + 2\lambda(1-\lambda)\left[\cos\theta - 1\right]$$

$$= 1 - 2\lambda(1-\lambda)2\sin^2\theta/2 \qquad \theta = \frac{\nu h}{L}$$

$$= 1 - \underbrace{\left[4(\lambda)(1-\lambda)\right]}\sin^2\theta/2$$

$0 \le \uparrow \le 1$ so long as $0 \le \lambda \le 1$

CFL $\Rightarrow$ all eigenvalues lie in region of abs. stab.

$\vec{u}_{j+1} = B\vec{u}_j \qquad B = I - \lambda A_p$

eigenvalues are $1 - \lambda(1 - e^{I\theta})$

Same analysis shows eigenvalues of B have $|K| \le 1$

if $\lambda \in [0,1]$.

(if not, eigenvectors grow at each timestep $1+\varepsilon$

and $(1+\varepsilon)^M$ is large if M is.

What happens to these eigenvectors from one timestep to next?

$$\lambda = R e^{I\phi}$$

$$\lambda \cdot e^{Irx} = R e^{I(rx+\phi)} = R e^{Ir(x+\phi/r)}$$

So in one time step the mode 1) scales by $|R|$

2) shifts to right by $-\phi/r$

what's supposed to happen?

1) scale by 1

2) shift to right by ak

$R = \sqrt{1 - 4\lambda(1-\lambda)\cos^2\theta} \approx 1 - 2\lambda(1-\lambda)\cos^2\theta$ if θ small

$\theta = rh$

$= \frac{2\pi k h}{L}$

so modes (we resolve well) are being damped.

(like the heat equation!)

$$\tan\phi = \frac{-\lambda \sin\theta}{1-\lambda+\lambda\cos\theta} \qquad \phi = \arctan\left(\underline{\qquad} \right)$$

$$\phi = -\lambda\theta\left[1 - \frac{1}{6}(1-\lambda)(1-2\lambda)\theta^2 + \cdots\right]$$

$$\frac{-\phi}{r} = \frac{ak}{h}\frac{\theta}{r}\left[\underline{\qquad}\right]$$

$$= ak\left[1 - \cdots\right]$$

wrong speed, by a factor of $1 - c\theta^2$

(but if $\lambda = \frac{1}{2}$, this term vanishes).