Last class

$$u_t + a u_x = 0 \qquad a>0, \text{ const}$$

$$u(x,0) = u_0(x) \qquad 0 \le x \le x_1$$

$$0 \le t \le T$$

$$u(0,t) = 0$$

Discretization by method of lines

$$\vec{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ \vdots \\ u_N(t) \end{bmatrix} \qquad u_i(t) \approx u(x_i, t)$$

$$x_i = ih$$

$$A = \begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & -1 & \ddots & \\ 0 & & -1 & 1 \end{bmatrix} \qquad u_x(x_i,t) \approx \frac{u(x_i,t) - u(x_{i-1},t)}{h}$$

$$\frac{d}{dt}\vec{u}(t) = -\frac{a}{h} A u(t)$$

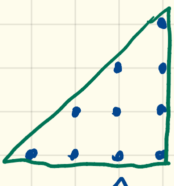
Now discretize in time (Forward Euler)

$$\vec{u}_{j+1} = \vec{u}_j - \underbrace{\frac{ka}{h}}_{\lambda} A \vec{u}_j \qquad (\text{not } \nu)$$

$$= (I - \lambda A)\vec{u}_j \qquad (*)$$

Did some experiments and saw that if $a = 1$, it appeared we needed ($T = 1$, $x_0 = 1$) $M > N$ for stability.

$$u_{i,j+1} = (1-\lambda)u_{i,j} + \lambda u_{i-1,j} \qquad (u_{0,j} = 0)$$



$u_{i,j}$

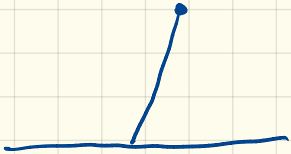
← depends on

$j=0$

triangle is the "numerical domain of dependence" of $u_{i,j}$

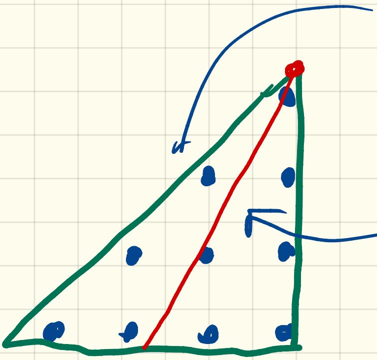
The values at $u_{i,j}$ depend only on these spots.

True domain of dependence is the characteristic curve,



to the past up to initial time. (if r.h.s isn't 0, can depend on stuff along this line)

Reasonable restriction: the true domain of dependence should lie in the numerical D.o.D

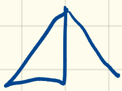


$\frac{k}{h}$

$$\frac{k}{h} \leq \frac{1}{a}$$

$$0 \leq \frac{ak}{h} \leq 1$$

otherwise



We get the latter by assuming $a > 0$.

This condition is known as the CFL condition

Courant, Friedrichs, Lewy '28

Note $\frac{ak}{h} = \lambda$ in

$$\vec{u}_{j+1} = (I - \lambda A)\vec{u}_j$$

If $T=1$, $x_1 = 1$, $a = 1$

then $k = \frac{1}{M}$ $h = \frac{1}{N}$

$$\frac{ak}{h} = \frac{N}{M} \le M \implies M \ge N.$$

If $a < 0$ we can't use this method; see text to see instability that arises

Instead, use a right derivative

$$\hat{A} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$u = 0$ at $x = x_1$

$$\vec{u}_{j+1} = \left(I - \frac{ak}{h} \hat{A} \right)$$

$$\therefore \quad \frac{1}{a} \le -\frac{k}{h}$$

$$1 \ge -\frac{ak}{h}$$

So $\quad 0 \le -\frac{ak}{h} \le 1 \quad$ if $\quad a < 0.$

Method: upwind.

Proof of convergence.

$$\frac{u(x_i, t_j+k) - u(x_i, t_j)}{k} = u_t(x_i, t_j) + u_{tt}(x_i, \hat{t}_j)k$$

$$\frac{u(x_i, t_j) - u(x_i - h, t_j)}{h} = u_x(x_i, t_j) + u_{xx}(\hat{x}_i, t_j)(h)$$

$$\text{LTE}: \left(u_t - a u_{xx}\right)\Big|_{x_i, t_j} + \left[u_{tt}(x_i, \hat{t}_j)k + a\, u_{xx}(\hat{x}_i, t_j) h \right]$$

$$\tau_{ij} = O(k) + O(h)$$

assuming u_{tt}, u_{xx} exist + cts.

$$u_{i,j+1} = (1-\lambda)\, u_{i,j} + \lambda\, u_{i-1,j}$$

weighted average if $0 \le \lambda \le 1$

$\lambda = 0$ $\quad u_{i,j+1} = u_{i,j}$

$\lambda = 1$ $\quad u_{i,j+1} = u_{i-1,j}$

$\frac{ak}{h}$ $\quad a = 0 \iff \lambda = 0$

$\frac{ak}{h}$ $\quad a = \frac{h}{k} \iff \lambda = 1$

we get an exact solution

(and we saw this!)

If a depends on space, can't pick $\lambda = 1$ everywhere.

Notation $\quad e_{i,j} = u_{i,j} - u(x_i, t_j)$

$$\tau_j = \max_i |\tau_{i,j}|$$

$$\tau = \max_j \tau_j$$

$$E_j = \max_i |e_{i,j}|$$

$$e_{i,j+1} = (1-\lambda) e_{i,j} + \lambda e_{i+1,j} - k\tau_{i,j}$$

$$|e_{i,j+1}| \leq |(1-\lambda) e_{i,j}| + |\lambda e_{i+1,j}| + k\tau$$

$0 \leq \lambda \leq 1$

$$\leq (1-\lambda) |e_{i,j}| + \lambda |e_{i-1,j}| + k\tau$$

$$\leq (1-\lambda) E_j + \lambda E_j + k\tau$$

$$= E_j + k\tau$$

$$E_{j+1} \leq E_j + k\tau$$

$E_1 \le E_0 + k|\tau|$

$E_2 \le E_0 + 2k|\tau|$

$E_M \le E_0 + Mk|\tau|$

$= E_0 + T|\tau|$

So if $E_0 = 0$ $\quad E_M = O(|\tau|) = O(h) + O(k)$

assuming two continuous second derivatives,

which is plainly violated in our example....

assuming $0 \le \lambda \le 1$.