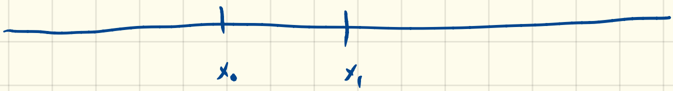
Hyperbolic PDEs

Consider $u(x,t)$ a density (stuff / length)



$$U(t) = \int_{x_0}^{x_1} u(x,t)\,dx$$ total stuff between x_0, x_1 at t.

$$\frac{dU}{dt} = -F(x_1,t) + F(x_0,t)$$ fluxes at endpoints, to right
(units: stuff per time)

Hypothesis: There is a velocity $v(x,t)$ and stuff is moving with prescribed velocity v.

$$F(x,t) = \underset{\uparrow}{v}(x,t) \cdot u(x,t)$$

$$\frac{\text{length}}{\text{time}} \cdot \frac{\text{stuff}}{\text{length}} = \frac{\text{stuff}}{\text{time}} \checkmark$$

$v > 0 \Rightarrow$ to the right.

$$\frac{d}{dt} \int_{x_0}^{x_1} u(x,t)\,dx = -v(x_1,t)\,u(x_1,t) + v(x_0,t)\,u(x_0,t)$$

$x_1 = x_0 + h$

$$\frac{1}{h} \int_{x_0}^{x_0+h} u_t(x,t)\,dx = -\frac{w(x_0+h,t) - w(x_0,t)}{h}$$

$w = vu$

Now $h \to 0$

$$u_t = -\partial_x (vu)$$

$$u_t + \partial_x (vu) = 0$$

just the flux in here.

Goes by a few names "advection equation"

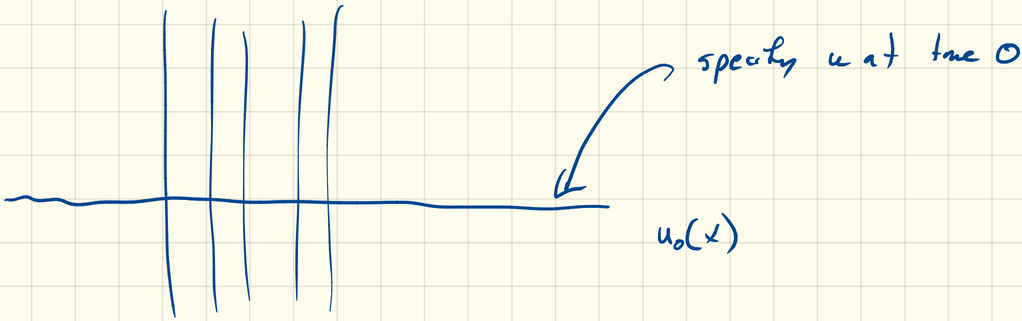
"transport equation".

We'll start with easy case: $v = a$, constant.

Super easy case: $a = 0$.

$$\partial_t u = 0$$

So u depends only on x.



$$u(x,t) = u_0(x)$$

Now if a is constant

$$u_t + a u_x = 0$$

Are there curves along which u is constant?

$$\gamma(r) = (x(r), t(r))$$

$$\frac{d}{dr} u(\gamma(r)) = u_x \frac{dx}{dr} + u_t \frac{dt}{dr}$$

If 1) $\frac{dt}{dr} = 1$

2) $\frac{dx}{dr} = a$

$$\Rightarrow \frac{d}{dr} u(\gamma(r)) = u_t + a u_x = 0.$$

$$\gamma(r) = (ar + x_0, r + t_0)$$

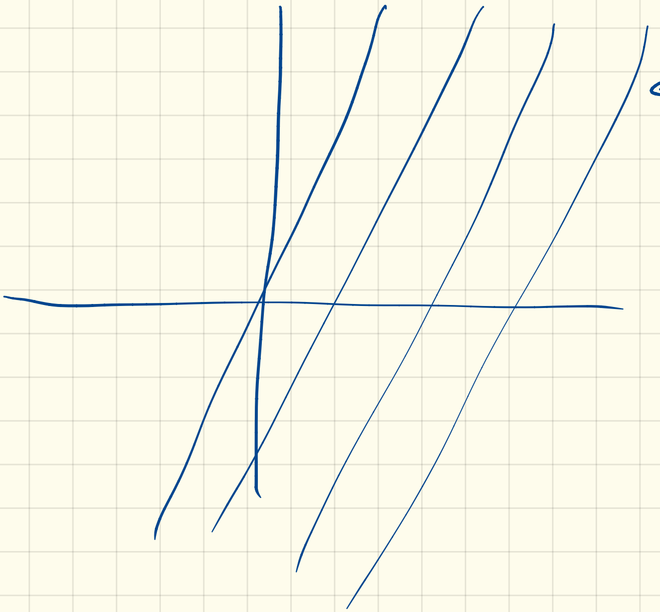
$x = ar + x_0$

$t = r + t_0$

$$\Rightarrow x = a(t - t_0) + x_0$$

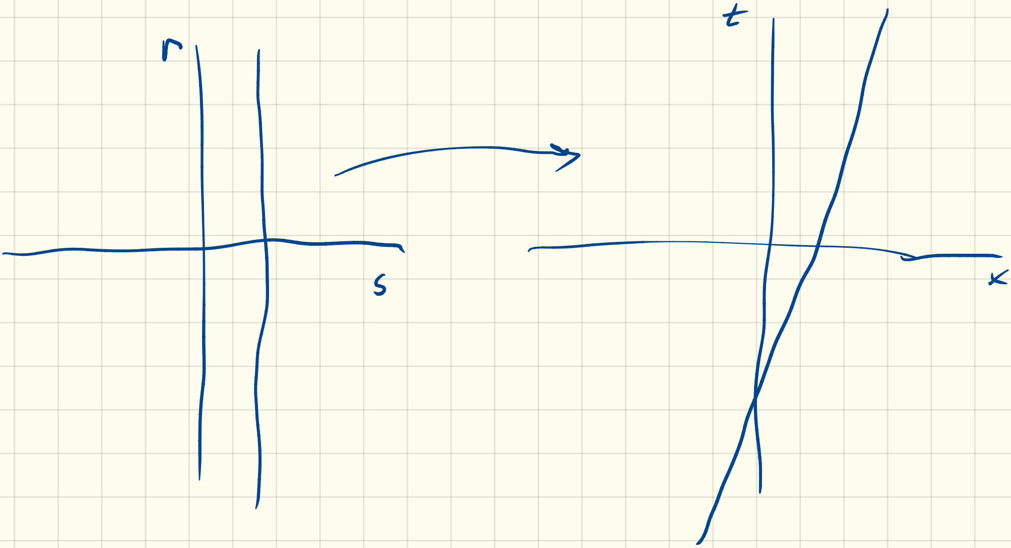
$$t = \frac{1}{a}(x - x_0) + t_0$$

These are lines with slope $\frac{1}{a}$.



Travel on this line and u is constant

New coordinates

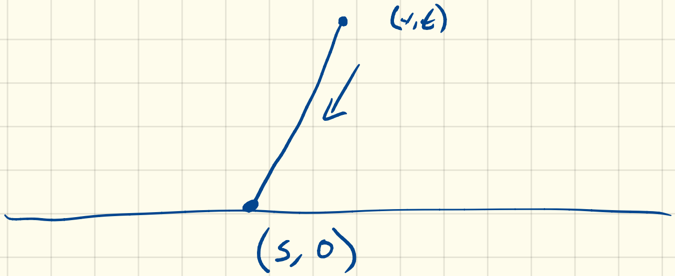


$$U(s,r) = u(x(s,r), t(s,r))$$

$$\frac{dU}{dr} = u_x \frac{\partial x}{\partial r} + u_t \frac{\partial t}{\partial r} = a u_x + u_t = 0$$

So U is a function of s alone.

We'll know u at $t=0$, $u(x,0) = u_0(x)$.



x, t come from (s,r) if we start at $(s,0)$ and follow γ up to parameter r and land on x, t.

$x_0 = s \quad t_0 = 0$

$$x = ar + s$$
$$t = r$$

$$U(s,r) = u(x(s,r), t(s,r))$$

$$u(x,t) = U(s(x,t), r(x,t))$$

$$U(s,0) = u(s,0) = u_0(s)$$
$$U(s,r) = u_0(s)$$

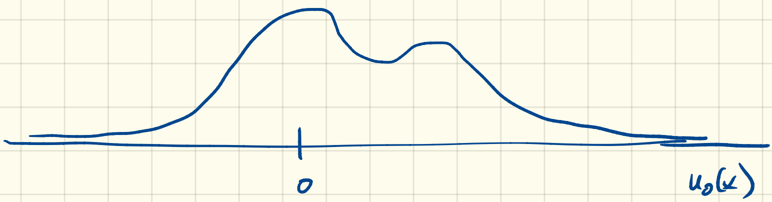
$$u(x,t) = u_0(s(x,t))$$

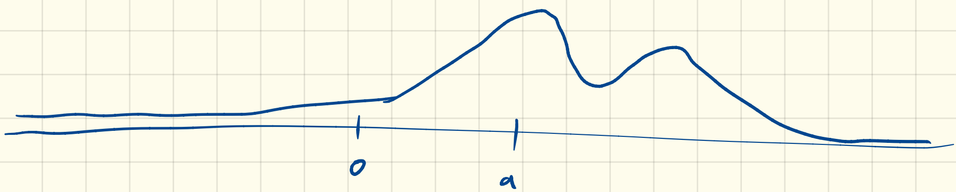
Can we invert? Sure $\quad s = x - ar = x - at$

$$u(x,t) = u_0(x - at)$$

Terminology: $\gamma(r)$: characteristic curves of the PDE

The method of solving by looking at the solution along the characteristics is the "method of characteristics"





$$u(x,1) = u_0(x-a)$$

The bump moves to the right with speed a if $a > 0$.

What if $$u_t + au = x$$

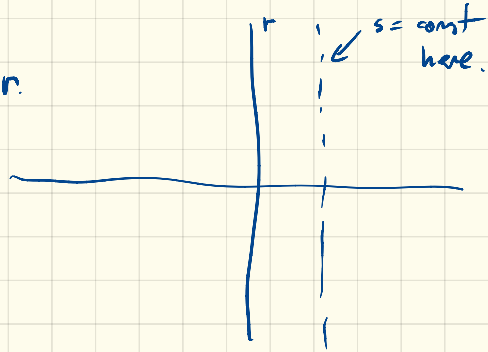
$$\partial_r U = ar + s$$

$$U = U(s,0) + \frac{ar^2}{2} + sr$$

$$u(x,t) = U(s(x,t),0) + \frac{ar(x,t)^2}{2} + sr$$

$$= u_0(x) + \frac{at^2}{2} + t(x - at)$$

We're just integrating in r.



$$u_t + (vu)_x = 0$$

$$u_t + v u_x = \underbrace{- u v_x}_{\text{ignore for now}}$$

$$u_t + v(x,t) u_x = 0$$

Same strategy $\quad \gamma(r) = (x(r), t(r))$

Suppose $\quad \gamma'(r) = \Big(v(x(r), t(r)), 1 \Big)$

$$x' = v(x,t)$$

$$t' = 1$$

$$\downarrow$$

$$t = r + t_0$$

$$\frac{dx}{dr} = v(x, r + t_0)$$

$$\frac{d}{dr}\, u\circ\gamma = u_t + u_x \frac{dx}{dt} = u_t + u_x v = 0$$

$$x_0 = s \qquad t_0 = 0$$

$$\left.\begin{aligned} x' &= v(x,t) \\ x(0) &= s \end{aligned}\right] \text{ ODE to solve}$$

$$U(s,r) = u(x(s,r),\, t(s,r))$$

$$\partial_r U = 0$$

$$\begin{aligned} U(s,r) = U(s,0) &= u(x(s,0),\, t(s,0)) \\ &= u(s,0) \\ &= u_0(s) \end{aligned}$$

$$u(x(s,r),\, t(s,r)) = u_0(s)$$

$$u(x,t) = u_0(s(x,t))$$

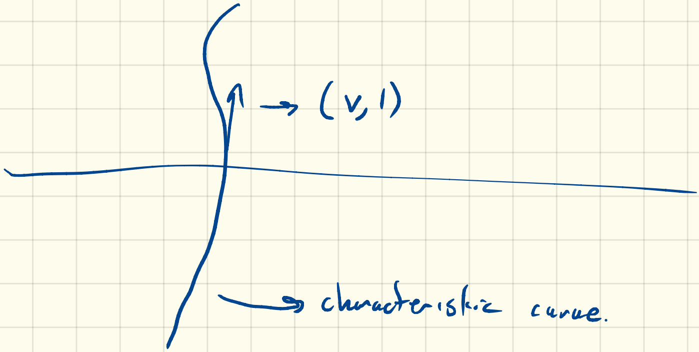
Hard to do in practice

1) Need to solve the ODE

2) Obtain $x(s,r)$, $t(s,r)$

But want $s(x,t)$!

But idea is clean



solution is constant on it.

And what of

$$u_t + (vu)_x = 0?$$

$$u_t + v u_x = -v_x u$$

$$U_r = -v_x\big(x(s,r), t(s,r)\big)\, U(s,r)$$
$$U(s,0) = u_0(s)$$

s is constant. This is an ODE to solve for U with s as a parameter.