Last class:

Want to establish convergence of θ-methods assuming only $\lambda(2\theta-1) \le \frac{1}{2}$

rather than $\lambda 2\theta \le 1$.

We'll do this using a different notion of convergence

vector norms

$$\|u\|_p = \left[\sum_{k=1}^{n} |v_k|^p\right]^{1/p} \qquad 1 \le p < \infty$$

$$\|u\|_\infty = \max_{k=1..n} |v_k|$$

$$B_p(r) = \{x : \|x\|_p \le r\}$$

matrix norms

$$A(B_p(1)) = \{A(x) : \|x\|_p \le 1\}$$
$$= \{Ax : x \in B_p(1)\}.$$

$$\|A\|_p = \inf_{r \ge 0} \quad A(B_p(1)) \subseteq B_p(r)$$

If $x \in B_p(1)$,

$\| Ax \|_p \leq \|A\|_p$, and $\|A\|_p$ is the smallest number that works for all $x \in B_p(1)$.

Alternatively,

$$\|A\|_p = \sup_{\|x\| \leq 1} \| Ax \|_p$$

$$\left(= \max_{\|x\| \leq 1} \|Ax\|_p \right)$$

It's the largest amount of stretching.

Now if $\|x\|_p < 1$, $x \neq 0$

$$\| Ax \| = \| A \tfrac{x}{\|x\|_p} \|_p \|x\|_p$$

$$\leq \| A \underbrace{\tfrac{x}{\|x\|_p}}_{\text{unit vector.}} \|_p$$

(make Ax bigger by going to a unit vector)

So $\|A\|_p = \sup_{\|x\| = 1} \| Ax \|_p$ $(*)$ (def 1)

But if $x \neq 0$

$$\frac{\|Ax\|_p}{\|x\|_p} = \left\| A \frac{x}{\|x\|_p} \right\|_p \leq \|A\|_p$$

$$\sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p.$$

But
$$\sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \geq \sup_{\|x\|=1} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p=1} \|Ax\|_p$$
$$= \|A\|_p.$$

So $\|A\|_p = \sup_{x \neq 0} \dfrac{\|Ax\|_p}{\|x\|_p}$ (def 2)

Key inequality:

$\|Ax\|_p \leq \|A\|_p \|x\|_p$ Obvious if $x = 0$.

Otherwise $\dfrac{\|Ax\|_p}{\|x\|} \leq \|A\|_p$ so

Exercise: $\|cA\|_p = |c| \, \|A\|_p$

Exercise: $\|A\|_p = 0 \iff A = 0$

Exercise: It is known that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

$0 \leq p \leq \infty$.

Use this to show

$$\|A+B\|_p \leq \|A\|_p + \|B\|_p.$$

(Triangle inequality)

For those in the know, this shows that $\|\cdot\|_p$ is a norm on $n \times n$ matrices.

Claim $\|AB\|_p \leq \|A\|_p \|B\|_p$.

Pf: Let x be a unit vector ($\|x\|_p = 1$).

Then $\|ABx\|_p \leq \|A\|_p \|Bx\|_p \leq \|A\|_p \|B\|_p \|x\|_p$

$= \|A\|_p \|B\|_p$.

Thus $\sup\limits_{\|x\|_p=1} \|ABx\|_p \leq \|A\|_p \|B\|_p$.

Here's the plan

$$B\hat{u}_{j+1} = A\bar{u}_j + k\vec{f}$$

We're going to show $\|B^{-1}A\|_2 \leq 1$

$\|B^{-1}\|_2 \leq 1$.

assuming

$\lambda(2\theta-1) \leq 1/2$

$$U_{j+1} = B^{-1}A\vec{U}_j + kB^{-1}\vec{f}$$

$$u_{j+1} = B^{-1}A\hat{u}_j + kB^{-1}\vec{f} + kB^{-1}\tau_j$$

$$E_{j+1} = B^{-1}A(E_j) - kB^{-1}\tau_j$$

$$\|E_{j+1}\|_2 \leq \|B^{-1}AE_j\|_2 + k\|B^{-1}\tau_j\|_2$$

$$\leq \|B^{-1}A\|_2\|E_j\|_2 + k\|B^{-1}\|\|\tau_j\|_2$$

$$\|E_{j+1}\|_2 \le \|E_j\|_2 + k\,\|\tau_j\|_2$$

$$\|\tau\|_{2,\infty} = \max_j \|\tau_j\|_2$$

$$\begin{aligned}
\|E_j\|_2 &\le \|E_0\|_2 + kj\,\|\tau\|_{2,\infty} \\
&\le \|E_0\|_2 + kM\,\|\tau\|_{2,\infty} \\
&= \|E_0\|_2 + T\,\|\tau\|_{2,\infty}
\end{aligned}$$

If $\|E_0\| = 0$,

$$\max_j \|E_j\|_2 \le T\,\|\tau\|_{2,\infty}$$

N entries, sum.

Now $\|\tau\|_{2,\infty} \to 0$ $\qquad \left[\sum (x_i)^2\right]^{1/2}$

$$\begin{aligned}
\|\tau_j\|_2 &= \left[\sum_{i=1}^{N} (\tau_{ij})^2\right]^{1/2} \\
&\le \max_i |\tau_{ij}| \left[\sum_{i=1}^{N} 1\right]^{1/2} = \sqrt{N}\,\|\tau_j\|_\infty
\end{aligned}$$

So $\quad \frac{1}{\sqrt{N}} \|\tau_j\|_2 \leq \|\tau_j\|_\infty$

$$\frac{1}{\sqrt{N}} \|\tau\|_{2,\infty} \leq \|\tau\|_\infty$$

$$\frac{1}{\sqrt{N}} \|E_j\|_2 \leq \frac{1}{\sqrt{N}} \|\tau\|_{2,\infty} \leq \|\tau\|_\infty$$

$$\max_j \frac{1}{\sqrt{N}} \|E_j\| \leq \|\tau\|_\infty$$

So if $\|\tau\|_\infty \to 0$ (consistency!) then $\frac{1}{\sqrt{N}} \|E\|_{2,\infty} \to 0$.

This is a weaker norm:

$$x = \overbrace{\left(N^{1/4}, 0, \ldots, 0 \right)}^{\to N \text{ entries}}$$

$$\|x\|_2 = N^{1/4}$$

$$\frac{\|x\|_2}{\sqrt{N}} = \frac{1}{N^{1/4}} \to 0.$$

But $\|x\|_\infty \not\to 0$.

This is a <u>weaker</u> notion of convergence.

We used to have $\|E\|_\infty \to 0$.

Now only $\frac{1}{\sqrt{N}} \|E\|_{2,\infty} \to 0$.

Error can concentrate on individual grid points, but on average, $\to 0$.

IOU: $\|B^{-1}A\|_2 \leq 1$ if $\lambda[2\theta - 1] \leq \frac{1}{2}$

$\|B^{-1}\| \leq 1$

Fact from linear algebra: every symmetric matrix A admits an orthonormal basis of eigenvectors.

$v_1, \ldots, v_n$ orthonormal

$\lambda_1, \ldots, \lambda_n$

$$P = [v_1, \ldots, v_n]$$

$$A = P \Lambda P^{-1} \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

$$P \Lambda P^{-1} v_k = P \Lambda e_k = P \lambda_k e_k = \lambda_k v_k.$$

$A v_k = \lambda_k v_k$ So they agree on a basis.

Such a P is called an orthogonal matrix, satisfies

$$P^T P = I$$

Note: If P is orthogonal, so is P^{-1} $(= P^T)$.

$$(P^{-1})^T P^{-1} = (P^T)^T P^{-1} = P P^{-1} = P^{-1} P = P^T P = I.$$

Lemma: If P is orthogonal, $\|P\|_2 = 1$.

Pf: Let $x \in \mathbb{R}^n$, $\|x\| = 1$. $\|Px\|_2 = \left[Px \cdot Px \right]^{1/2}$

$$= \left[x^T P^T P x \right]^{1/2} = \|x\|_2 = 1.$$

So $\sup_{\|x\|=1} \|Px\|_2 = 1$.

Lemma: If $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$,

$$\|A\|_2 = \max |\lambda_i|$$

Pf: Let $x = x_1 e_1 + \cdots + x_n e_n$. So $\|x\|_2 = \left[\sum |x_i|^2 \right]^{1/2}$.

Then $\|Ax\|_2 = \| \lambda_1 x_1 e_1 + \cdots + \lambda_n x_n e_n \|_2$ and

$$\|Ax\| = \left[\sum | \lambda_i x_i |^2 \right]^{1/2} \leq M \left(\sum |x_i|^2 \right)^{1/2}$$

$M = \max |\lambda_i|$. So $\|Ax\|_2 \leq M$.

But by taking $x = e_k$, $\|A\|_2 \geq \max(|\lambda_i|)$.

Def: $\max |\lambda_i| = \sigma(A)$, the spectral radius of A.

Prop: If A is symmetric, $\|A\|_2 = \sigma(A)$.

Pf: Write $A = P \Lambda P^{-1}$ where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

Observe $\|A\|_2 \leq \|P\| \, \|\Lambda\| \, \|P^{-1}\|$

$$= 1 \cdot \sigma(A) \cdot 1$$

$$= \sigma(A).$$

Now let x be an eigenvector of unit length.

Then $\|Ax\|_p = \|\lambda_k x\|_p = |\lambda_k| \, \|x\|_p = |\lambda_k|$.

So $\|A\|_p \geq \max |\lambda_k| = \sigma(A)$.

Now back to $B u_{j+1} = A u_j + k f_j$

Claim $B^{-1}A$ is symmetric.

A, B both are.

So is B^{-1}: $(B^{-1})^T = (B^T)^{-1} = B^{-1}$.

Moreover $B^{-1}A = A B^{-1}$ since A, B have a common basis of eigenvectors.

Now $(B^{-1}A)^T = A^T (B^{-1})^T = A B^{-1} = B^{-1} A$.

$$\| B^{-1} A \|_2 = \sigma(B^{-1}A)$$

$$\| B^{-1} \|_2 = \sigma(B)$$

Last class: eigenvalues of B: $\dfrac{1}{1 + 4(1-\theta)\lambda \sin^2(\nu h/2)}$ $\quad \nu = n\pi$

So $\sigma(B) \leq 1$.

Also

eigenvalues of $B^{-1}A$: $\dfrac{1-4\theta\lambda\sin^2(\pi h/2)}{1+4(1-\theta)\lambda\sin^2(\pi h/2)}$

each $\lambda_k \leq 1$.

$\sigma(B^{-1}A) \leq 1$ if each $\lambda_k \geq -1$.

$$-1-4(1-\theta)\lambda\sin^2(\pi h/2) \leq 1-4\theta\lambda\sin^2(\pi h/2)$$

$$4\lambda(2\theta-1)\sin^2(\pi h/2) \leq 2$$

$$\lambda(2\theta-1)\sin^2\left(\frac{\pi h}{2}\right) \leq \frac{1}{2}.$$

So if $\lambda(2\theta-1) \leq \frac{1}{2}$ then $\sigma(B^{-1}A) \leq 1$

and $\|B^{-1}A\|_2 \leq 1$.