Last class:

Introduced θ method:

$$u_{j+1} = u_j + \lambda(1-\theta) D\, u_{j+1} + \lambda \theta D u_j$$

$\theta = 1$: Forward Euler
$\theta = 0$: Backward Euler
$\theta = 1/2$: Crank-Nicholson

Von Neumann analysis:

$$\lambda[2\theta - 1] \leq \frac{1}{2} \qquad \text{for "stability."}$$

Always true for $\theta \leq 1/2$. Otherwise a restriction on λ.

For $\theta = 1$ $\lambda \leq 1/2$, same restriction as before.

Convergence proof assuming $1 - 2\lambda\theta \geq 0$

$$\max_{i,j} \left| U_{i,j} - u(x_i, t_j) \right| \longrightarrow 0$$

Two conditions

a) $\lambda(2\theta - 1) \leq 1/2$ for Von Neumann stability

b) $\lambda 2\theta \leq 1$ for convergence

Exercise: if $0 \leq \theta \leq 1$

λ satisfies b) $\Rightarrow$ λ satisfies a)

Can we get convergence only assuming a) ?

Yes, but we have to relax our idea of what convergence means.

Why care? $\lambda \leq \frac{1}{2\theta}$ $k \leq h^2 \left(\frac{1}{2\theta}\right)$

Have $k \sim h^2$ (but $\frac{1}{2\theta} \to \infty$ as $\theta \to 0$, so restriction is less severe as $\theta \to 0$)

$\theta = 0$, no restriction on λ, lovely.

But error still will be $O(k) + O(h^2)$

$\theta = \frac{1}{2}$ we'll have error is $O(k^2) + O(h^2)$

Our max principle convergence proof requires

$\lambda \leq 1$ $\quad k \leq h^2$ so still need

$O(h^2)$ timesteps.

But, if we have convergence for $\lambda(2\theta - 1) \leq \frac{1}{2}$,

then we get $O(k^2) + O(h^2)$ error and arbitrary λ,

a big win. (Can take $k \sim h$ and see $O(h^2)$).

If you let me change how I measure error I can get convergence in case $\theta \le 1/2$.

$$\left[I - (1-\theta)\lambda D\right]\vec{u}_{j+1} = \left[I + \theta\lambda D\right]\vec{u}_j + k\vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j$$

Let's talk about the eigenvalues/vectors of A, B.

For D, $u_i = \sin(r x_i)$ $(0 \le x \le 1)$

$1 \le i \le N$

$\lambda = -4\sin^2\left(\frac{rh}{2}\right)$ $r = k\pi$

$1 \le k \le N$

Eigenvalues of D : $-4\sin^2(rh/2)$ $\quad r = n\pi \quad n = 1, \dots, N$

Eigenvalues of $\theta\lambda D$: $-4\theta\lambda\sin^2(rh/2)$ (same evecs)

Eigenvalues of $A = I + \theta\lambda D$: $1 - 4\theta\lambda\sin^2(rh/2)$

Eigenvalues of $B = I - (1-\theta)\lambda D$: $1 + 4(1-\theta)\lambda\sin^2(rh/2)$

Eigenvalues of B^{-1} : $(1 + 4(1-\theta)\dots\dots)^{-1}$

Eigenvalues of $B^{-1}A$: $\dfrac{1 - 4\theta\lambda\sin^2(rh/2)}{1 + 4(1-\theta)\lambda\sin^2(rh/2)}$

Finally: $B^{-1}A = AB^{-1}$: why? same evecs!

Evals of B are all ≥ 1. So B is invertible.

(B not inv $\Leftrightarrow$ 0 is an eigenvalue)

$$\vec{U}_{j+1} = B^{-1} A \vec{U}_j + B^{-1} k \vec{f}_j$$

$$u_{j+1} = B^{-1} A u_j + B^{-1} k f_j + B^{-1} k \tau_j$$

$$E_{j+1} = B^{-1} A E_j - B^{-1} \tau_j k$$

Vector norms $\qquad v = (v_1, \dots, v_n)$

$$\|v\|_1 = \sum_{k=1}^{n} |v_k|$$

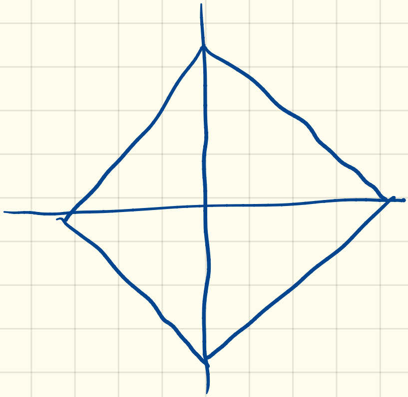
$$\|v\|_2 = \left[\sum_{k=1}^{n} |v_k|^2 \right]^{1/2}$$

$$\|v\|_p = \left[\sum_{k=1}^{n} |v_k|^p \right]^{1/p} \qquad 0 \le p < \infty$$

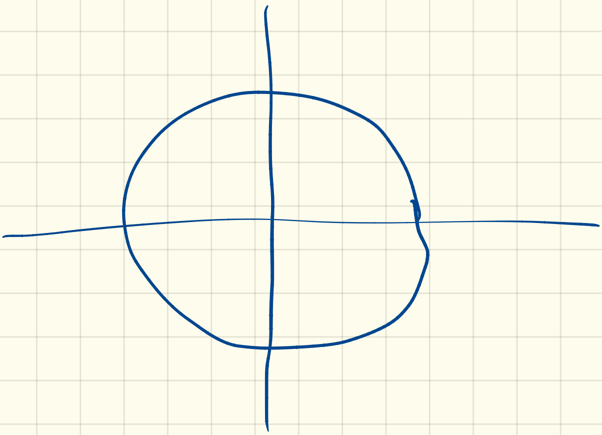
$$\|v\|_\infty = \max_k |v_k| \qquad \|v\|_p \longrightarrow \|v\|_\infty$$

as $p \to \infty$.

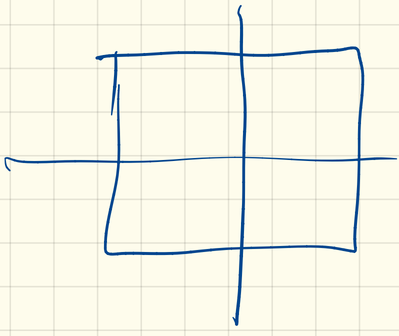
$\|v\|_1 = 1$



$\|v\|_2 = 1$



$\|v\|_\infty = 1$



Matrix norms:

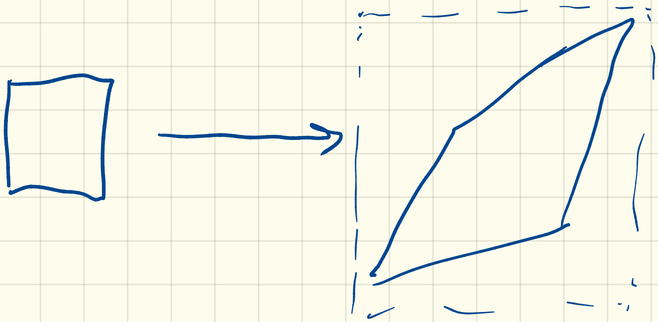
A : $n \times n$ matrix

$B_p(r) = \{ x : \|x\|_p \leq r \}$ ball of radius r w.r.t. p-norm.

$\|A\|_p$: $A(B_p(1)) = \{ Ax : x \in B_p(1) \}$

$$\|A\|_p = \inf_r : A(B_p(1)) \subseteq B_p(r))$$

It's a measure of the amount of stretching A does, using the $\|\cdot\|_p$ to quantify stretching.



$$\|Ax\|_p \le \|A\|_p \, \|x\|_p$$

(if $x=0$, trivial. Otherwise $\dfrac{x}{\|x\|_p}$ is a unit vector

$$\left\| A\left(\frac{x}{\|x\|_p}\right)\right\| \le \|A\|_p \quad \text{and}$$

$$\left\| A\left(\frac{x}{\|x\|_p}\right)\right\| = \frac{\|Ax\|}{\|x\|_p}$$

In fact $\|A\|_p$ is the smallest number n such that

$$\|Ax\|_p \le n\,\|x\|_p.$$

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} \quad (1)$$

$$= \sup_{\|x\|_p=1} \frac{\|Ax\|_p}{\|x\|_p}$$

$$= \sup_{\|x\|_p=1} \|Ax\|_p \quad (2)$$

either are defs.

Exercise: $\|cA\|_p = |c| \, \|A\|_p$

Exercise: $\|A\|_p = 0 \iff A = 0$

Exercise: It is known that $\|x+y\|_p \leq \|x\|_p + \|y\|_p$

$0 \leq p \leq \infty$.

Use this to show

$$\|A+B\|_p \leq \|A\|_p + \|B\|_p.$$

(Triangle inequality)

For those in the know, this shows that $\|\cdot\|_p$ is a norm on $n \times n$ matrices.

Exercise: $\|AB\|_p \leq \|A\|_p \, \|B\|_p$.

Here's the plan

$$B\hat{u}_{j+1} = A\bar{u}_j + k\vec{f}$$

We're going to show $\|B^{-1}A\|_2 \leq 1$ assuming

$\lambda(2\theta-1) \leq 1/2$

$\|B^{-1}\|_2 \leq 1$.

$$U_{j+1} = B^{-1}A\vec{U}_j + kB^{-1}\vec{f}$$

$$u_{j+1} = B^{-1}A\hat{u}_j + kB^{-1}\vec{f} + kB^{-1}\tau_j$$

$$E_{j+1} = B^{-1}A(E_j) - kB^{-1}\tau_j$$

$$\|E_{j+1}\|_2 \leq \|B^{-1}A E_j\|_2 + k\|B^{-1}\tau_j\|_2$$

$$\leq \|B^{-1}A\|_2 \|E_j\|_2 + k\|B^{-1}\| \|\tau_j\|_2$$

$$\|E_{j+1}\|_2 \le \|E_j\|_2 + k\,\|\tau_j\|_2$$

$$|\tau|_2 = \max_j \|\tau_j\|_2$$

$$\begin{aligned}\|E_j\|_2 &\le \|E_0\|_2 + kj\,|\tau|_2 \\ &\le \|E_0\|_2 + kM\,|\tau|_2 \\ &= \|E_0\|_2 + T\,|\tau|_2\end{aligned}$$

If $\|E_0\| = 0$,

$$\max_j \|E_j\|_2 \le T\,|\tau|_2$$

N entries, grow.

Now $|\tau|_2 \not\to 0$ $\qquad \left[\sum (x_i)^2\right]^{1/2}$

$$\max_j \left(\frac{1}{\sqrt{N}} \|E_j\|_2 \right) \le T \underbrace{\frac{1}{\sqrt{N}} |\tau|_2}_{\to 0}$$

So if $|\tau|_\infty \to 0$ (consistency!)

then $\max_j \frac{1}{\sqrt{N}} \|E_j\|_2 \to 0$

This is a weaker norm:

$$x = (N^{1/4}, 0, \ldots, 0)$$

$$\|x\|_2 = N^{1/4}$$

$$\frac{\|x\|_2}{\sqrt{N}} = \frac{1}{N^{1/4}} \to 0.$$

But $\|x\|_\infty \not\to 0$.

This is a <u>weaker</u> notion of convergence.

IOU: $\|B^{-1}A\|_2 \le 1$ if $\lambda[2\theta-1] \le \frac{1}{2}$

$\|B^{-1}\| \le 1$

Fact from linear algebra: every symmetric matrix A admits an orthonormal basis of eigenvectors.

$v_1, \dots, v_n$ orthonormal

$\lambda_1, \dots, \lambda_n$

$P = [v_1, \dots, v_n]$

$A = P \Lambda P^{-1}$ $\qquad \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$P \Lambda P^{-1} v_k = P \Lambda e_k = P \lambda_k e_k = \lambda_k v_k$.

$A v_k = \lambda_k v_k$ So they agree on a basis.

$P^T P = I \Rightarrow P^{-1} = P^T$

$\|P\|_2 = 1$

$\|\Lambda\|_2 = \max(|\lambda_1|, \dots, |\lambda_n|)$

$\|P^{-1}\|_2 = 1$

spectral radius.
$\sigma(A)$

$\|P \Lambda P^{-1}\|_2 \le \overbrace{\max(|\lambda_1|, \dots, |\lambda_n|)}$

In fact $\|P\Lambda P^{-1}\|_2 = \max(|\lambda_1|, \dots, |\lambda_n|) = \sigma(A)$.

Just use an eigenvector:

$$\|Ax\|_p = \|\lambda x\|_p = |\lambda| \, \|x\|_p$$

So $\|A\|_p \geq |\lambda|$.

Prop: If A is symmetric, $\|A\|_2 = \sigma(A)$.

B, A are symmetric

B^{-1} is also symmetric.

$$B^{-1} = (B^T)^{-1} = (B^{-1})^T$$

$$(B^{-1}A)^T = A^T (B^{-1})^T$$
$$= A B^{-1}$$
$$= B^{-1} A \quad (A, B \text{ have a common basis of eigenvectors})$$

eigenvalues of $B^{-1}A$:

$$\frac{1 - 4\theta\lambda \sin^2(rh/2)}{1 + 4(1-\theta)\lambda \sin^2(rh/2)} \qquad r = n\pi \quad 1 \le n \le N$$

$$-1\left[1 + 4(1-\theta)\lambda s\right] \le 1 - 4\theta\lambda s$$

$$4\lambda s\left[2\theta - 1\right] \le 2$$

$$\lambda s\left[2\theta - 1\right] \le \frac{1}{2}$$

$$\lambda\left[2\theta - 1\right] \le \frac{1}{2}$$

$\sigma(B^{-1}A) \le 1$ assuming $\lambda\left[2\theta - 1\right] \le 1/2$

$\|B^{-1}A\|_2 \le 1.$