Generalization: Θ-method

$$\vec{u}_{j+1} = \vec{u}_j + \theta \lambda D \vec{u}_j + (1-\theta)\lambda D \vec{u}_{j+1} + \vec{f}_j$$

$$\left[1 - (1-\theta)\lambda D\right]\vec{u}_{j+1} = \left[1 + \theta\lambda D\right]\vec{u}_j + \vec{f}_j$$

explicit when $\theta = 1$. Backwards Euler, $\theta = 0$.

$$q\left[-(1-\theta)\lambda e^{-Jrh} - (1-\theta)\lambda e^{Jrh}\left[1+2(1-\theta)\lambda\right]\right]e^{Jrx_i}$$

$$= \left[\theta\lambda\left(e^{-Jrh} + e^{Jrh}\right) + \left[1 - 2\theta\lambda\right]\right]e^{Jrx_i}$$

$$q = \frac{1 - 4\theta\lambda \sin^2\left(\frac{rh}{2}\right)}{1 + 4(1-\theta)\lambda\sin^2\left(\frac{rh}{2}\right)}$$

Want $-1 \le q \le 1$ for stability

But $g \le 1$ always (top ≤ 1, bottom ≥ 1)

$$-1 \le \frac{1 - 4\theta\lambda \sin^2(rh/2)}{1 + 4(1-\theta)\lambda \sin^2(rh/2)}$$

$$-1 - 4(1-\theta)\lambda \sin^2(rh/2) \le 1 - 4\theta\lambda \sin^2(rh/2)$$

$$4\lambda \underbrace{\sin^2\left(\frac{rh}{2}\right)}_{\text{crank up to } 1} \left[\theta - (1-\theta)\right] \le 2$$

$$\lambda\left[2\theta - 1\right] \le \frac{2}{4} = \frac{1}{2}$$

If $0 \le \theta \le \frac{1}{2}$ $\lambda[2\theta - 1] \le 0$, so always satisfied.

Exercise: All these methods are $O(k) + O(h^2)$.

But in fact for $\theta = \frac{1}{2}$ is $O(k^2) + O(h^2)$

$\hookrightarrow$ Crank Nicholson

Explicit: If $\frac{k}{h^2} \geq \frac{1}{2}$, no guarantee

$\theta = 1$

If $\frac{k}{h^2} \leq \frac{1}{2}$, error is $O(k) + O(h^2)$

$\theta < \frac{1}{2}$ Error is $O(k) + O(h^2)$

Still require $k \sim h^2$ to keep error from time discretization dominating that from space. constant for $\sim$ depends on θ, improves as $\theta \to 1/2$.

$\theta = \frac{1}{2}$ Error is $O(k^2) + O(h^2)$

Now need $k \sim h$

to keep time discretization from dominating.

Convergence of θ methods (first pass)

Based on maximum principle $\quad (u_{0,j}=0,\ u_{N+1,j}=0)$

$$(1+2\lambda(1-\theta))\,u_{i,j+1} = \lambda(1-\theta)\left[u_{i-1,j+1}+u_{i+1,j+1}\right]$$

$$+\lambda\theta\left[u_{i-1,j}+u_{i+1,j}\right]$$

$$+\left[1-2\lambda\theta\right]u_{i,j}$$

We will assume $\quad 1-2\lambda\theta \geq 0$ $\quad$ different from $\theta \leq \frac{1}{2}$!

$$\lambda\theta \leq \frac{1}{2}$$

$(\theta=0,$ always$)$

Coefficients on RHS are all ≥ 0, add to

$$2\lambda(1-\theta)+2\lambda\theta+1-2\lambda\theta$$

$$= 2\lambda - 2\lambda\theta + 2\lambda\theta + 1 - 2\lambda\theta$$

$$= 1 + 2\lambda - 2\lambda\theta$$

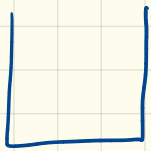
$$= 1 + 2\lambda(1-\theta)$$

which is the coeff on the left.

value here is an average of values

So value at an interior point can't beat its neighbors.

So max/min occur on



Now: Supposing $\lambda\theta \le \frac{1}{2}$ we prove convergence.

$E_{i,j} = U_{i,j} - u(x_i, t_j)$ as in convergence proof for forward Euler.

$$\left[1 + 2\lambda(1-\theta)\right] E_{i,j+1} = \lambda(1-\theta)\left[E_{i-1,j+1} + E_{i+1,j+1}\right]$$

$$+ \lambda\theta\left[E_{i-1,j} + E_{i+1,j}\right]$$

$$+ \left[1 - 2\lambda\theta\right] E_{i,j} - k\tau_{ij}$$

$E_j = \max_i |E_{i,j}|$

$$(1 + 2\lambda(1-\theta)) E_{j+1} \le 2\lambda(1-\theta) E_{j+1}$$

$$+ 2\lambda\theta E_j + (1 - 2\lambda\theta) E_j + k|\tau|$$

So

$$E_{j+1} \leq E_j + k|\tau|$$

$$E_j \leq E_0 + Mk|\tau|$$

$$= E_0 + T|\tau|$$

So, so long as $|\tau| \to 0$

$\max_j E_j \to 0$ also and we have

convergence.

$$x_j^{(n)}, t_j^{(n)} \to (x,t)$$

$$U_{i,j}^{(n)} \to u(x,t)$$

$$\max_{i,j} |U_{i,j} - u(x_i, t_j)| \longrightarrow 0$$

(kind of l_∞ convergence)

This gives convergence in case $\lambda\theta \leq \frac{1}{2}$

$$\frac{k}{h^2}\theta \leq \frac{1}{2}$$

progressively weaker conditions as $\theta \to 0$.

(Arbitrary step size for $\theta = 0$!)

If you let me change how I measure error I can get convergence in case $\theta \leq 1/2$.

$$\left[I - (1-\theta)\lambda D\right]\vec{u}_{j+1} = \left[I + \theta\lambda D\right]\vec{u}_j + \vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + \vec{f}$$

Eigenvalues of D are negative, so those of

$-(1-\theta)\lambda D$ are ≥ 0.

So those of $\underbrace{I - (1-\theta)\lambda D}_{B}$ are > 0.

So no kernel!