$$u_t = u_{xx} + f$$

$$u_{tt} = u_{txx} + f_t$$

$$= u_{xxt} + f_t$$

$$= u_{xxxx} + f_{xx} + f_t$$

$$|\tau| \leq \left[\frac{k}{2} + \frac{h^2}{6}\right] \max |u_{xxxx}| + \frac{k}{2}\left[\max |f_t| + \max |f_{xx}|\right]$$

$u_{tt} = u_{xxxx} + f_t + f_{xx}$ is, alas, a compatibility condition on the initial data:

$u_{tt} = 0$ on boundary.

If $f = 0$,

$u_{xxxx} = 0$ on boundary is needed.

$$C = \max |u_{xxxx}| + \max |f_t| + \max |f_{xx}|$$

$$|\tau| \leq \left[\frac{k}{2} + \frac{h^2}{6}\right] C$$

$U_{i,j}$ numerical solution

$$U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i+1,j} + kf_{i,j}$$

$$u_{i,j} = u(x_i, t_j)$$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + kf_{i,j} + k\tau_{i,j}$$

$$E_{i,j} = U_{i,j} - u_{i,j}$$

$$E_{i,j+1} = \lambda E_{i-1,j} + (1-2\lambda)E_{i,j} + \lambda E_{i+1,j} - k\tau_{i,j}$$

$$E_j = \max_i |E_{i,j}|$$

$$|E_{i,j+1}| \leq |\lambda|\,|E_{i-1,j}| + |1-2\lambda|\,E_{i,j} + |\lambda|\,|E_{i+1}| + k|\tau_{i,j}|$$

use $\lambda \leq \frac{1}{2}$

$$\leq \lambda E_j + (1-2\lambda)E_j + \lambda E_j + k|\tau_{i,j}|$$

$$E_{j+1} \leq E_j + k|\tau| \qquad |\tau| = \max_{i,j} |\tau_{i,j}| \leq C\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

$$E_1 \le E_0 + k\tau$$

$$E_2 \le E_1 + k\tau$$

$$\le E_0 + 2k\tau$$

$$\vdots$$

$$E_M \le E_0 + \underbrace{Mk}_{T}\tau$$

$$E_j \le E_0 + TC\left[\frac{k}{2} + \frac{h^2}{6}\right]$$

$$0 \le j \le M.$$

$$\max_j E_j \le TC\left[\frac{k}{2} + \frac{h^2}{6}\right]$$

Thm: If $h_{(n)} \to 0$, $k_{(n)} \to 0$ $\frac{k}{h^2} \le \frac{1}{2}$, + smooth (2 time, 4 space derivatives continuous)

$$\left(x_i^{(n)}, t_j^{(n)}\right) \to (x, t)$$

$$\Rightarrow \quad u_{i,j}^{(n)} \longrightarrow u(x_i, t_j)$$

Since we have issues with transients, it will be helpful to work with an A-stable scheme.

Backwards Euler:

$$\vec{u}_{j+1} = \vec{u}_j + \frac{k}{h^2} D \vec{u}_{j+1} + k\vec{f}_j$$

$$\left[I - \lambda D \right] \vec{u}_{j+1} = \vec{u}_j + k \vec{f}_j$$

This is a linear system to solve for $\vec{u}_{j+1}$.

But $A = (I - \lambda D)$ is tri-diagonal

$$-\lambda \quad 1+2\lambda \quad -\lambda$$
$$\ddots$$

So solving is $O(n)$.

Fourier Analysis: $w_i = e^{Jrx_i}$

$u_{ij} = q^j w_i$ ← assume solution like this

$$q\left[-\lambda e^{-Jhr} + (1+2\lambda) - \lambda e^{Jhr}\right] e^{Jx_i r} = e^{Jx_i r}$$

$$\begin{aligned} q^{-1} &= \left[1+2\lambda - \lambda 2\cos(hr)\right] \\ &= \left[1 + 2\lambda\left(1-\cos(hr)\right)\right] \\ &= \left[1 + 4\lambda \sin^2(hr/2)\right] \end{aligned}$$

$$q = \frac{1}{1+4\lambda\sin^2\left(\frac{hr}{2}\right)} \leq 1, \text{ regardless of } \lambda.$$

We'll shortly see this method is

1) convergent, regardless of λ

2) $O(h) + O(k^2)$.

Generalization: Θ-method

$$\vec{u}_{j+1} = \vec{u}_j + \Theta\lambda D\vec{u}_j + (1-\Theta)\lambda D\vec{u}_{j+1} + \vec{f}_j$$

$$\left[1 - (1-\theta)\lambda D\right]\vec{u}_{j+1} = \left[1 + \Theta\lambda D\right]\vec{u}_j + \vec{f}_j$$

explicit when $\Theta = 1$. Backwards Euler, $\Theta = 0$.

$$q\left[-(1-\theta)\lambda e^{-Jrh} - (1-\theta)\lambda e^{Jrh}\left[1+2(1-\theta)\lambda\right]\right]e^{Jrx_i}$$

$$= \left[\Theta\lambda\left(e^{-Jrh}+e^{Jrh}\right) + \left[1-2\theta\lambda\right]\right]e^{Jrx_i}$$

$$q = \frac{1 - 4\Theta\lambda\sin^2\left(\frac{rh}{2}\right)}{1 + 4(1-\theta)\lambda\sin^2\left(\frac{rh}{2}\right)}$$

Want $-1 \le q \le 1$ for stability