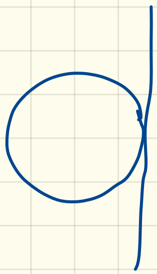
Last class:

Thought of explicit method as Euler's method applied to

$$u' = \frac{1}{h^2} D x$$

Expect eigenvalues of $\frac{1}{h^2} D \sim -\pi^2 n^2 \quad 1 \le n \le N$

Thus region of abs. stability is important



$$k\lambda \geq -2$$

$$-\pi^2 k N^2 \geq -2$$

$$k N^2 \leq \frac{2}{\pi^2}$$

$$\sim \frac{k}{h^2} \leq \frac{2}{\pi^2}$$

So $\underbrace{\frac{k}{h^2}}_{\lambda} < \frac{2}{\pi}$ ish.

This analysis is only heuristic: we don't yet know the eigenvalues of

$$\frac{1}{h^2} D.$$

To learn these, it's enough to study D.

We make a lucky guess

$$w_j = e^{Jrx_j} \qquad J^2 = -1$$

(makes trig id's easy).

$2 \le j \le N-1$

$$Dw_j = e^{Jrx_j}\left[e^{-Jrh} - 2 + e^{Jrh}\right]$$

$$= -2\, w_j \left[1 + \cos(rh)\right]$$

$$= -4\, w_j \left[\sin^2\left(\frac{rh}{2}\right)\right]$$

$x_{j+1} = x_j + h$

$e^{J\theta} = \cos\theta + J\sin\theta$

$e^{-J\theta} = \cos\theta - J\sin\theta$

So we almost have an eigenvalue: analysis doesn't apply at $j=1$, $j=N$.

But we let

$$v_j = \operatorname{Im}(w_j) \qquad r = n\pi$$

$$v_0 = 0 \qquad \left(h = \frac{1}{N+1}\right)$$
$$v_{N+1} = 0$$

$$\operatorname{Im}(D w_j) = D \operatorname{Im}(w_j) = D v_j$$

$$\operatorname{Im}(D w_j) = \operatorname{Im}\left(-\cancel{2}4\sin^2\left(\tfrac{rh}{2}\right) w_j\right) = -\cancel{2}4\sin^2\left(\tfrac{rh}{2}\right) \underbrace{\operatorname{Im}(w_j)}_{v_j}$$

So $\vec{v}_n$ is an eigenvector of D with e-value $-2\sin^2\left(\frac{r_n h}{2}\right)$

$$r_n = n\pi \qquad 1 \leq n \leq N.$$

Sweet.

Eigenvalues of $\frac{1}{h^2} D$ are $\displaystyle -\frac{4}{h^2}\sin^2\left(\frac{r_n h}{2}\right)$

If $\frac{r_n h}{2}$ is small, $\displaystyle \sim -\frac{4}{h^2}\left[\frac{n^2\pi^2}{4} h^2\right] = -n^2\pi^2$

(nh is small)

Analysis from Euler's method:

$$k\left(-\frac{4}{h^2}\right)\sin^2\left(\frac{r_n h}{2}\right) > -2$$

$$\frac{k}{h^2}\underbrace{\sin^2\left(\frac{r_n h}{2}\right)}_{\text{no control.}} < \frac{1}{2}$$

So $\frac{k}{h^2} < \frac{1}{2}$

If $k \geq \frac{h^2}{2}$ then the time step is too long for the fastest transient modeled with this number of spatial steps.

Fourier Analysis: (a fast rule of thumb approach)

$$\vec{u}_{j+1} = (I + \lambda D)\, u_j$$

$$v_k = e^{J r x_k}$$

For all but the boundary points

Suppose $u_j = v$

$$u_{i,j+1} = \lambda v_{i-1} + (1-2\lambda) v_i + \lambda v_{i+1}$$

$$= v_i \left[(1-2\lambda) + \lambda e^{-Jrh} + \lambda e^{Jrh} \right]$$

$$= v_i \left[1 + \lambda \left[-2 + \cos(rh) \right] \right]$$

$$= v_i \left[1 - 4\lambda \sin^2(rh) \right]$$

amplification factor

So $u_{j+1} = [1 - 4\lambda \sin^2(rh)]\, u_j$ except at boundary points.

To avoid instability want

$$-1 \le 1 - 4\lambda \sin^2(rh) \le 1$$

$$\frac{1}{2} \ge \lambda \sin^2(rh) \Rightarrow \frac{1}{2} \ge \lambda$$

same condition.

As time progresses, solution oscillates + grows if $\lambda > \frac{1}{2}$

Note: this condition is compatible with maximum principle.

Let's prove convergence assuming $\lambda \leq \frac{1}{2}$.

Specifically, $h_n \to 0$, $k_n \to 0$, $\frac{k_n}{h_n^2} \leq \frac{1}{2}$ $\forall n$.

I'd like to be a little more careful with LTE

$$\text{I} \quad \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} = u_t(x_i, t_j) + \frac{k}{2} u_{tt}(x_i, \eta_j)$$

$$t_j \leq \eta_j \leq t_j + h$$

$$\text{II} \quad \frac{u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)}{h^2} = u_{xx}(x_i, t_j) + \frac{h^2}{6} u_{xxxx}(\hat{\eta}_i, t_j)$$

$$x_{i-1} \leq \hat{\eta}_i \leq x_{i+1}$$

$$\underbrace{\text{I} + \text{II} - \underbrace{f(x_i, t_j)}_{(u_t - u_{xx})(x_i, t_j)}}_{\tau} = \frac{k}{2} u_{tt}(x_i, \eta_j) + \frac{h^2}{6} u_{xxxx}(\hat{\eta}_i, t_j)$$

$$u_t = u_{xx} + f$$

$$u_{tt} = u_{txx} + f_t$$

$$= u_{xxt} + f_t$$

$$= u_{xxxx} + f_t$$

$$|\tau| \leq \left[\frac{k}{2} + \frac{h^2}{6}\right] \max|u_{xxxx}| + \frac{k}{2}\max|f_t|$$

$u_{tt} = u_{xxxx} + f_t$ is, alas, a compatibility condition on the initial data:

$u_{tt} = 0$ on boundary.

If $f = 0$,

$u_{xxxx} = 0$ on boundary is needed.

$$C = \max|u_{xxxx}| + \max|f_t|$$

$$|\tau| \leq \left[\frac{k}{2} + \frac{h^2}{6}\right] C$$