Last class:

Derived explicit method for solving heat equation.

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j} + f_{i,j}$$

$$\lambda = \frac{k}{h^2}$$

But we had observed instability: as time advances we saw spurious wiggles show up in every example we looked at.

Today's goal: why did this happen?

I'm going to write

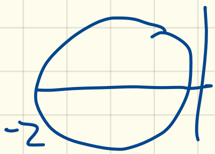
$$\vec{u}_{j+1} = (I + \lambda D)\vec{u}_j + \vec{f}_j \quad \text{where}$$

$$D = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & & \\ 1 & -2 & 1 & 0 & \cdots & 0 & \\ 0 & 1 & -2 & 1 & 0 & \cdots & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

This is Euler's method applied to ODE system

$$\vec{u}\,' = \frac{1}{h^2} D \vec{u} + \vec{f}$$

Region of absolute stability for Euler's method:




$\frac{1}{h^2} D$ is supposed to be an approximation of $\partial_x^2$

with eigenvalues $-(\pi n)^2 \quad 1 \le n \le N$

these get more negative as $N$ grows.

So we expect $\quad -(\pi n)^2 k > -2$

$$k < \left[\frac{2}{\pi}\right] \frac{1}{n^2}$$

$$k < \frac{2}{\pi} \frac{1}{N^2} \simeq \frac{2}{\pi} h^2$$

So $\underbrace{\frac{k}{h^2}}_{\lambda} < \frac{2}{\pi}$ ish.

This analysis is only heuristic: we don't yet know the eigenvalues of

$$\frac{1}{h^2} D.$$

To learn these, it's enough to study $D$.

We make a lucky guess

$$w_j = e^{Jrx_j} \qquad J^2 = -1 \qquad \text{(makes trig id's easy)}.$$

$2 \le j \le N-1$

$$Dw_j = e^{Jrx_j}\left[e^{-Jrh} - 2 + e^{Jrh}\right] \qquad x_{j+1} = x_j + h$$

$$= -2w_j\left[1 + \cos(rh)\right] \qquad e^{J\theta} = \cos\theta + J\sin\theta$$

$$e^{-J\theta} = \cos\theta - J\sin\theta$$

$$= -4w_j\left[\sin^2\left(\frac{rh}{2}\right)\right]$$

So we almost have an eigenvalue: analysis doesn't apply at $j=1$, $j=N$.

But we let

$$v_j = \operatorname{Im}(w_j) \qquad r = n\pi$$

$$v_0 = 0 \qquad \left(h = \frac{1}{N+1}\right)$$
$$v_{N+1} = 0$$

✓

$$\operatorname{Im}(Dw_j) = D\operatorname{Im}(w_j) = D v_j$$

$$\operatorname{Im}(Dw_j) = \operatorname{Im}\left(-\cancel{2}4\sin^2\left(\frac{rh}{2}\right)w_j\right) = -\cancel{2}4\sin^2\left(\frac{rh}{2}\right)\underbrace{\operatorname{Im}(w_j)}_{v_j}$$

So $\vec{v}_n$ is an eigenvector of $D$ with e-value $-2\sin^2\left(\frac{r_n h}{2}\right)$

$$r_n = n\pi \qquad 1 \leq n \leq N.$$

Sweet.

Eigenvalues of $\frac{1}{h^2}D$ are $-\frac{4}{h^2}\sin^2\left(\frac{r_n h}{2}\right)$

If $\frac{r_n h}{2}$ is small, $\sim -\frac{4}{h^2}\left[\frac{n^2\pi^2}{4}h^2\right] = -n^2\pi^2$

($nh$ is small)

$$E_1 \leq E_0 + k\tau$$

$$\begin{aligned} E_2 &\leq E_1 + k\tau \\ &\leq E_0 + k\tau + k\tau \\ &= E_0 + 2k\tau \\ &\vdots \end{aligned}$$

$$\begin{aligned} E_M &\leq E_0 + Mk\tau \\ &= E_0 + T\tau \end{aligned}$$

So $E_0 \to 0$, $\tau \to 0 \longrightarrow E \to 0$

* so long as * $\quad 1 - 2\lambda \geq 0$

$$1 \geq 2\lambda$$

$$\frac{1}{2} \geq \lambda.$$