Past due to start PDEs!

Ch 3: diffusion problems.

Model: heat equation.

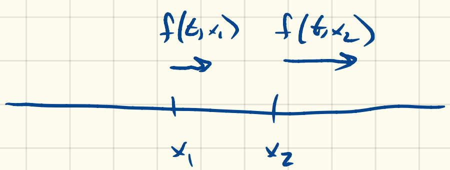
space domain $[0,1]$ (imagine a rod)

u: a density of some kind (particles, energy, heat ~ temp)

$u(t,x)$

Flux $f(t,x)$ tells you at time t, at position x, the rate at which stuff is passing by, to the right, in units $[u/t]$

$$\frac{d}{dt}\int_{x_1}^{x_2} u(t,x)\,dx = f(t,x_1) - f(t,x_2)$$



Leveling hypothesis

$$f(t,x) \sim u_x$$

$$f(t,x) = -k\, u_x \qquad \text{(more generally, } k(t,x)\text{, see later)}$$

$$\frac{d}{dt}\int_{x_1}^{x_2} u(t,x)\,dx = -k\,u_x(t,x_1) + k\,u_x$$

$$= k\int_{x_1}^{x_2} u_{xx}(t,x)\,dx$$

$$\int_{x_1}^{x_2} \left[u_t - u_{xx}\right](t,x)\,dx = 0$$

ind of x_1, x_2. So $u_t - u_{xx} = 0$.

Domain

$$\Omega = [0,T] \times [0,1]$$

Exercise: If $u_t - k u_{xx} = g$

interpret g. Hint: what are its units?

Exercise If $k(t,x)$

$$u_t - \partial_x \left(k(t,x)\, u_x \right) = 0.$$

We'll take $k=1$ even though this hides the units. (can arrange by scaling time)

Boundary conditions: (every PDE has its own reasonable classes of BC's).

For us $u(0,x) = u_0(x)$ (Initial distribution).

+ Conditions at $x=0$, $x=1$

Dirichlet: $u(t,0)$ $u(t,1)$ prescribed.

akin to maintaining fixed temps at end, no matter what.

Neumann: u_x prescribed at $x=0$, $x=1$.

(flux is $-ku_x$, so we are prescribing flux)

We can mix at either end, of course.

Robin: $u_x - cu = 0$ flux is a function of u.

$-ku_x = kcu$

We'll focus for now on homogeneous dirichlet conditions $u\big|_{0,1} = 0$.

$u_t = u_{xx}$ can be thought of as an analog of

$u_t = Au$, a linear system of ODEs.

If $Av = \lambda v$ then there's a solution

$u = e^{\lambda t} v$

$u_t = \lambda u$ ✓

$Au = \lambda u$ ✓

If A is diagonalizable with eigen pairs $(v_1, \lambda_1), \ldots, (v_n, \lambda_n)$

$$u = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n \quad \text{is a solution.}$$

For initial data u_0, express it

$$u_0 = c_1 v_1 + \cdots + c_n v_n.$$

Then $u(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n$

solves $\begin{aligned} u' &= Au \\ u(0) &= u_0 \end{aligned}$

Caution: not every matrix is diagonalizable: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not. $u = e^{At} u_0$ $\quad e^B = \sum_{j=0}^{\infty} \frac{B^j}{j!}$ solves.

At any rate, what's our analog for eigenvectors?

$$Au = u_{xx}$$

$$\begin{aligned} u_{xx} &= \lambda u \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned} \quad + \text{ BC's}$$

(This is why we introduced homogeneous conditions)

$u_{xx} = \lambda u$ depends on sign of λ

$e^{\pm\sqrt{\lambda}\, x}$ $\quad \lambda \geq 0$

$\cos(\sqrt{-\lambda}\, x)$ $\quad \sin(\sqrt{-\lambda}\, x)$ $\quad \lambda < 0$

But to set $u(0) = 0$, $u(1) = 0$

only $\lambda < 0$ works with $u = \sin(k\pi x)$

eigenfunction $\quad \lambda = -k^2\pi^2$

$$\underbrace{e^{-k^2\pi^2 t} \sin(k\pi x)}_{\text{solution of heat equation.}}$$

$k=1$ $\quad$ $k=2$ $\quad$ $k=3$

decay: $-\pi^2$ $\quad$ $-4\pi^2$ $\quad$ $-9\pi^2$

A $\quad u = \sum_{k=1}^{n} c_k e^{-k^2\pi^2 t} \sin(k\pi x)$ solves PDE, BC's

with initial cond $\sum_{k=1}^{n} c_k \sin(k\pi x)$.

Morally, one would like to start with any u_0, and write

$$u_0 = \sum_{k=1}^{\infty} a_k \sin(k\pi x)$$

the sum to ∞ makes this subtle.

What does "=" mean?

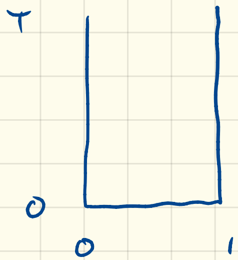
One hopes

$$u = \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t} \sin(k\pi x)$$ solves the PDE.

Finding conditions to justify this procedure is the domain of Fourier analysis, which is too far afield.

Maximum principle for heat equation:

"under the forward flow in time heat can't concentrate"



$\Omega = [0,1] \times [0,T]$

$\partial\Omega$ is boundary

$\partial\Omega^*$ is boundary except for

$\{t=T,\ x\in(0,1)\}$

Weak maximum principle:

If $\quad u_t - u_{xx} \leq 0 \quad$ then $\displaystyle \max_{\Omega} u = \max_{\partial\Omega^*} u$.

Cor: if $\quad u_t - u_{xx} \geq 0 \quad$ then $\displaystyle \min_{\Omega} u = \min_{\partial\Omega^*} u$.

Cor: if $\quad u_t - u_{xx} = 0$, u achieves both its max and min on $\partial\Omega^*$

Cor: $\left.\begin{array}{r} u_t - u_{xx} = f \\ u|_{t=0} = u_0 \\ + \text{dirichlet BC's} \end{array}\right]$ has at most one solution; $v = u_1 - u_2$ has $\begin{array}{l} v_t - v_{xx} = 0 \\ v|_{\partial\Omega^*} = 0. \end{array}$

Pf: We first show the property holds if $u_t - u_{xx} < 0$ everywhere in interior.

At a point in $\Omega \setminus \partial\Omega^*$ where a max is achieved,

$u_t \geqslant 0$

← uses not at $t=0$

$u_x = 0$

] uses not on space boundary

$u_{xx} \leqslant 0$.

So $u_t - u_{xx} \geqslant 0$ at this point

But no such point exists.

Now suppose only $u_t - u_{xx} \leqslant 0$.

Let $v_\varepsilon = u - \varepsilon t$

So $(v_\varepsilon)_t - (v_\varepsilon)_{xx} = -\varepsilon + u_t - u_{xx} < 0$.

So v_ε achieves its max on $\partial\Omega^*$.

$$\left[\max_{\Omega\setminus\partial\Omega^*} u\right] - \varepsilon T \leq \max_{\Omega\setminus\partial\Omega^*} (u - \varepsilon t) \leq \max_{\partial\Omega^*} (u - \varepsilon t) \leq \max_{\partial\Omega^*} u$$

Now send $\varepsilon \to 0$.

Energy

$$E(t) = \frac{1}{2}\int_0^1 |u_x|^2 \, dx$$

$$\frac{d}{dt} E(t) = \int_0^1 u_x \, u_{xt} \, dx$$

$$= \int_0^1 \partial_x (u_x u_t) - u_{xx} u_t \, dx$$

$$= \int_0^1 \partial_x (u_x u_t) - (u_t)^2 \, dx$$

$$= u_x u_t \Big|_0^1 - \int_0^1 (u_t)^2 \, dx$$

Homogeneous Neumann $\Rightarrow \frac{d}{dt} E(t) \leq 0$

Homogeneous Dirichlet $\Rightarrow \frac{d}{dt} E(t) \leq 0$

Solution becomes "smoother"

If $E(t) = 0$ at some point, $E(t) \equiv 0$.

Exercise: Show that there is at most one solution (C^2, say, in domain).

$$u_t = u_{xx}$$

$$u(0,x) = u_0$$

$$u(t,0) = b_0(t)$$
$$u(t,1) = b_1(t)$$

$$\left[\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t\right]^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} t^n$$

$$\sum \frac{\lambda^n t^n}{n!} = e^{\lambda t} \qquad \sum_{n=0}^{\infty} \frac{n\lambda^{n-1} t^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} t^n$$

$$= t \sum_{n=1}^{\infty} \frac{(t\lambda)^{n-1}}{(n-1)!}$$

$$= t e^{\lambda t}$$

$$\boxed{e^{tA} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}}$$