$$u_{n+1} = u_n + \lambda h u_n$$
$$\rho = (1+\lambda h)^n$$

Recap:

LMMs $\qquad \underbrace{\alpha_n u_{n+k} + \cdots + \alpha_0 u_k} = h\left(\beta_n u_{n+k} + \cdots + \beta_0 u_0\right)$

$$\sigma(\rho) = \alpha_n \rho^n + \cdots + \alpha_1 \rho + \alpha_0$$

characteristic poly.

Zero stable $\iff$ bounded initial data remain bounded as $h \to 0$

$\iff$ roots of $\sigma$ satisfy root condition ($|\rho| \le 1$, etc).

Dahlquist:

If consistent: zero stable $\iff$ convergent

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Absolute stability: Apply method to $u' = \lambda u$.

Absolutely stable for $z = h\lambda$ if solutions remain bounded as $n \to \infty$

Important if there are strong transients: want them to decay, even if we don't model them accurately.

$$\underbrace{\sigma(\rho) - z\,\eta(\rho)} = 0 \qquad \eta(\rho) = \beta_n \rho^n + \cdots + \beta_1 \rho + \beta_0$$

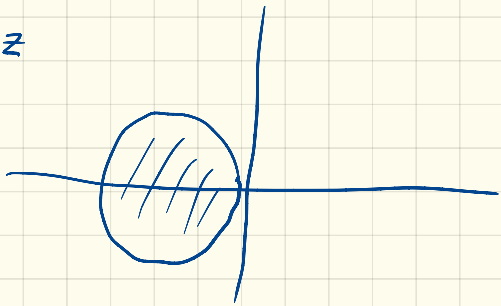
LMM stability polynomial

Absolutely stable for $z$ if all roots of
$q(\rho)=\sigma(\rho) - z\,\eta(\rho)$ have $|\rho| \le 1$ (+ simplicity).

Region of absolute stability:

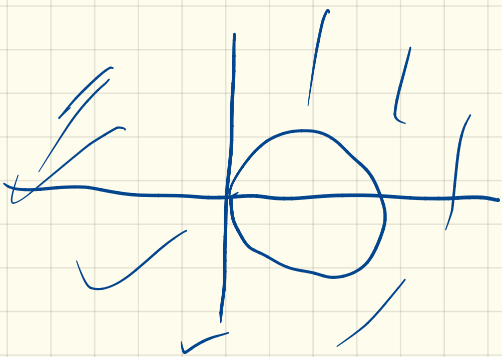
$\{ z : \text{method is abs stable for that } z \}$

Euler: $\rho - 1 = z$



Backward Euler $\rho - 1 = z\rho$

$\rho(1-z) = 1 \qquad \rho = \dfrac{1}{1-z}$



decay whenever $\mathrm{Re}\,\lambda < 0$,

$\boxed{\text{A-stable}}$

How to visualize more generally when you can't find the roots

$$\sigma(\rho) - z\,\eta(\rho) = 0$$

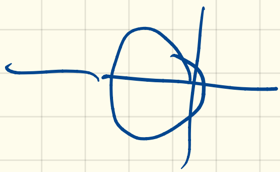
$$z = \frac{\sigma(\rho)}{\eta(\rho)}$$

Pick your favorite $\rho$. Plug into. For that $z$, your favorite $\rho$ will be a root of the stability polynomial.

$$f(\theta) = \frac{\sigma(e^{i\theta})}{\eta(e^{i\theta})}$$

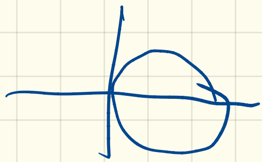
This will be the boundary of the absolute stability region.

Euler: $z = -1 + e^{i\theta}$



B. Euler $z = 1 - \frac{1}{\rho}$

$$= 1 - e^{-i\theta}$$



See HW for example with an Adams-Bashforth.

Abs stability for R-K (k-stage)

$$Y_1 = u_n + h \sum_{j=1}^{k} a_{1j} f(t + c_j h, Y_j)$$

$$\vdots$$

$$Y_k = u_n + h \sum_{j=1}^{k} a_{kj} f(t + c_j h, Y_j)$$

$Y_j$ an approximate solution at $t + c_j h$

$$u_{n+1} = u_n + h \sum_{j=1}^{k} b_j f(t + c_j h, Y_j)$$

Apply to $f(t,u) = \lambda u$

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} = u_n \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\to w} + h\lambda A \begin{bmatrix} Y_1 \\ \vdots \\ Y_k \end{bmatrix} \qquad A = [a_{kj}]$$

$$(I - zA) Y = u_n w$$

$$\uparrow$$

$Y = u_n f(z, b)$ for $z$ small

$$Y = u_n (I - zA)^{-1} w$$

$$\begin{aligned} u_{n+1} &= u_n + z\, b \cdot u_n (I - zA)^{-1} w \\ &= u_n \left[ 1 + z\, b \cdot (I - zA)^{-1} w \right] \\ &= R(z)\, u_n \end{aligned}$$

stability function for R-K.

will be a polynomial in $z$ if the method is explicit

Abs stable at $z$ if $|R(z)| \leq 1$.

So visualize the abs stability region by looking at contour $|R(z)| = 1$. See notebook. contourplot.